Exercise Series No. 04

Exercise 1: We endow $\mathbb{R}^+$ with the internal composition law defined by:

$$\forall x, y \in \mathbb{R}^+, \quad x * y = \sqrt{x^2 + y^2}.$$

1. Show that $*$ is commutative, associative, and admits a neutral element.

2. Determine the invertible elements.

Exercise 2: Show that $(G, *)$ is a group in the following cases and specify if it is abelian (commutative):

1. $x * y = \frac{x+y}{1+xy}, \quad$ on $G =]-1, 1[$;

2. $(x, y) * (x', y') = (x + x', ye^{x'} + y'e^{-x}), \quad$ on $G = \mathbb{R}^2$.

Exercise 4: Let $(G, \times)$ be a group. Prove that the following subsets are subgroups of G:

1. $C(G) = \{x \in G : \forall y \in G, xy = yx\}$, where $C(G)$ is called the center of G;

2. $aHa^{-1} = \{aha^{-1} : h \in H\}$, where $a \in G$ and H is a subgroup of G.

Exercise 5: Are the following applications group homomorphisms? If so, compute the kernel and the image, and deduce whether they are group isomorphisms or automorphisms.

1. $\varphi : (\mathbb{R}^*, \times) \to (\mathbb{R}^*, \times), \quad \varphi(x) = x^n$, $n \in \mathbb{N}^*$;

2. $\varphi : (\mathbb{R}, +) \to (\mathbb{C}^*, \times), \quad \varphi(t) = e^{2\pi i t}$.