POWER SERIES

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Power Series :

1) Definition :

We call power serie any serie of function $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

we write : $\sum_{n=0}^{\infty} a_n x^n = \sum_{n\geq 0} U_n$ And $(a_n)_{n\geq 0}$ is a real sequence.

$U_n = a_n x^n$ is called the general term of the serie$\{S_n\}$.

Examples : $\sum_{n\geq 0} x^n$, $\sum_{n\geq 0} \frac{n}{n+1} x^n$, $\sum_{n\geq 0} 3^n x^n$

2) Radius of convergence :

The radius of convergence of a power serie $\sum_{n=0}^{\infty} a_n x^n$

Is the positive real R such that :

$$\begin{cases} \text{If } |x| < R \text{ the serie } \{S_n\} \text{ is convergent.} \\ \text{If } |x| > R \text{ the serie } \{S_n\} \text{ is divergent.} \end{cases}$$

3) Domain of convergence : We call domain of convergence of a power serie $\{S_n\}$ the set of all reals where the serie is convergent.

If $|x| < R$, $\{S_n\}$ converges and the domain of convergence is the interval of the center zéro and the radius R . $D = \{ x \in \mathbb{R} \ / \ |x| < R\}$.

Theorem :

Let $\sum_{n\geq 0} a_n x^n$ a power serie. And R its radius of convergence. We have :

$$\begin{cases} R = 0 \iff D = \{0\} \\ R = +\infty \iff D = \mathbb{R} \\ 0 < R < +\infty \iff D =]-R \quad R[\end{cases}$$

Remark :

1) For $x = \pm R$ We don't conclude about convergence of the série$\{S_n\}$.

2) If the power serie is in the form $\sum_{n\geq 0} a_n (x - x_0)^n$ Then :

$$\begin{cases} R = 0 \iff D = \{x_0\} \\ R = +\infty \iff D = \mathbb{R} \\ 0 < R < +\infty \iff D = \left]-R + x_0 \quad R + x_0\right[\end{cases}$$

4) Tests of convergence : (Calculation of radius R)

Let $\sum_{n\geq 0} a_n x^n$ a power serie.

a) Test of Cauchy- Hadamard

If $\lim_{n\to+\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to+\infty} \sqrt[n]{|a_n|}\ |x| < 1$ $\left(\text{The serie } \sum_{n\geq 0} a_n x^n \text{ converges}\right)$.

The radius of convergence R is given by : $R = \frac{1}{l}$. with $l = \lim_{n\to+\infty} \sqrt[n]{|a_n|}$

.

Examples :

1) $\sum_{n\geq 0} (2x)^n$, $a_n = 2^n$; $\lim_{n\to+\infty} \sqrt[n]{2^n} = \lim_{n\to+\infty} 2 = 2$

Then : $R = \frac{1}{l} = \frac{1}{2}$.

For $x = \pm\frac{1}{2}$ the numerical series $\sum_{n\geq 0} 1$ and $\sum_{n\geq 0} (-1)^n$ are divergent.

Then the domain of convergence is $D = \left]-\frac{1}{2}\ \frac{1}{2}\right[$.

2) $\sum_{n\geq 1} \left(\frac{x}{n}\right)^n$ $\quad R = \frac{1}{l}$, $\quad l = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{1}{n}\right)^n} = \lim_{n\to+\infty} \frac{1}{n} = 0$

$R = +\infty$ and $D = \mathbb{R}$.

b) Test of D'Alembert :

If $\lim_{n\to+\infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| < 1$ $\left(\text{The serie } \sum_{n\geq 0} a_n x^n \text{ converges}\right)$.

Then the radius of convergence R given by : $R = \frac{1}{l}$. with $l = \lim_{n \to +\infty} \left|\frac{a_{n+1}}{a_n}\right|$

.

Examples : 1) $\sum_{n\geq 1} \frac{x^n}{n+1}$ $\qquad a_n = \frac{1}{n+1}$

$$\lim_{n \to +\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to +\infty} \frac{n+1}{n+2} = 1. \quad \text{Then : } R = 1.$$

For $x = 1$, $\sum_{n\geq 1} \frac{1}{n+1} \sim \sum_{n\geq 1} \frac{1}{n}$ diverges.

For $x = -1$, $\sum_{n\geq 1} \frac{(-1)^n}{n+1}$ converges. (According to Leibnitz)

Then : $D =]-1 \quad 1]$

2) $\sum_{n\geq 0} n!\, x^n$ $\qquad a_n = n!$

$$R = \frac{1}{l} \quad , \quad l = \lim_{n \to +\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to +\infty} \frac{(n+1)!}{n!} = \lim_{n \to +\infty} (n+1) = +\infty$$

Then : $R = \frac{1}{l} = 0$ And $D = \{0\}$.

5) Sum of the power serie :

Consider $\sum_{n\geq 0} a_n x^n$ a power serie of the radius R.

We call $S_n = \sum_{k=0}^{n} a_k x^k$ The partial sum of the serie $\sum_{n\geq 0} a_n x^n$.

We call sum of the serie $\sum_{n\geq 0} a_n x^n$ a limit of its partial sum .

Written : $S = \lim_{n \to +\infty} S_n$.

Example : $\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$, $a_n=\frac{1}{2^n}$, $R=\frac{1}{\lim_{n\to+\infty}\sqrt[n]{a_n}}=2$, $D=]-2\ \ 2[$

$S_n=\sum_{k=0}^{n}a_kx^k$ $(S_n)_n$ a geometric sequense of basic $q=\frac{x}{2}$.

$S_n=\frac{1-q^n}{1-q}=\frac{1-\left(\frac{x}{2}\right)^n}{1-\frac{x}{2}}$, The sum of the serie $\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$ is $S=\lim_{n\to+\infty}S_n=\frac{1}{1-\frac{x}{2}}=\frac{2}{2-x}$.

6) Derivation and integration of power series :

Theorem :

Consider $\sum_{n\geq 0}a_nx^n$ a power serie , with radius R and the sum S. Then :

(1) The series $\sum_{n\geq 1}na_nx^{n-1}$ and $\sum_{n\geq 0}\frac{a_n}{n+1}x^{n+1}$, obtained by derivation.

And integration of the serie $\sum_{n\geq 0}a_nx^n$, have the same radius of convergence.

(2) $S'(x)=\left(\sum_{n\geq 0}a_nx^n\right)'=\sum_{n\geq 1}na_nx^{n-1}$

(3) $\int_0^x S(t)dt=\int_0^x\left(\sum_{n\geq 0}a_nt^n\right)dt=\sum_{n\geq 0}\frac{a_n}{n+1}x^{n+1}$

Example : $S=\sum_{n\geq 2}\frac{x^n}{(n-1)n}$

$\sum_{n\geq 0}x^n=\lim_{n\to+\infty}\frac{1-x^{n+1}}{1-x}=\frac{1}{1-x}$ By integration we obtain :

$$\sum_{n\geq 0}\frac{x^{n+1}}{n+1}=-ln(1-x)\xrightarrow{m=n+1}\sum_{m\geq 1}\frac{x^m}{m}=-ln(1-x)\ \text{ By integration}$$

$$\sum_{m\geq 1}\frac{x^{m+1}}{(m+1)m}=-\int_0^x ln(1-t)dt=(1-x)\,ln(1-x)+x\xrightarrow{n=m+1}\sum_{n\geq 2}\frac{x^n}{n(n-1)}=(1-x)ln(1-x)+x$$

7) Power series expansion (developement) :

Consider $f: D\subseteq\mathbb{R}\to\mathbb{R}$ Derivable on D. and $x_0\in D$.

We call the serie $\sum_{n=0}^{+\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ The Taylor serie of f in the neighborhood of x_0 .

Theorem :

Any derivable function on the interval D is equal to the sum of a power serie converges in this intervale.

Examples : Developement (expansion) in the neighborhood of zero $(x_0=0)$.

1) $f(x)=e^x=f(0)+xf'(0)+\frac{x^2f''(0)}{2}+\cdots+\frac{x^nf^{(n)}(0)}{n!}+$

$=1+x+x^2+\cdots+x^n+\cdots=\sum_{n\geq 0}\frac{x^n}{n!}$.

2) $f(x)=\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+(-1)^nx^{2n+1}+\cdots=\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$

3) $f(x)=\cos(x)=\sum_{n\geq 0}\frac{(-1)^n}{(2n)!}x^{2n}$

4) $f(x)=\text{sh}(x)=\frac{1}{2}(e^x-e^{-x})$, $\quad g(x)=ch(x)=\frac{1}{2}(e^x+e^{-x})$

On a $\begin{cases} e^{x} = \sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots \\ e^{-x} = \sum_{n\geq 0} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^n}{n!}x^n + \cdots \end{cases}$

$$sh(x) = x + \frac{1}{3!}x^3 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + \cdots = \sum_{n\geq 0} \frac{1}{(2n+1)!}x^{2n+1}$$

$$ch(x) = 1 + \frac{1}{2}x^2 + \cdots + \frac{1}{(2n)!}x^{2n} + \cdots = \sum_{n\geq 0} \frac{1}{(2n)!}x^{2n}$$

5) $f(x) = ln(1+x)$

We have : $\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^n x^n + \cdots = \sum_{n\geq 0} (-1)^n x^n$

$$f(x) = ln(1+x) = \int_0^x \frac{du}{1+u} = \sum_{n\geq 0} \frac{(-1)^n}{n+1} x^{n+1}$$

Examples :

1) $f(x) = \frac{x+2}{x^2+2x-3} = \frac{1}{4}\left(\frac{3}{1-x} - \frac{1}{x+3}\right) = \frac{3}{4}\sum_{n\geq 0} x^n - \frac{1}{12}\frac{1}{1+\frac{x}{3}} = \frac{3}{4}\sum_{n\geq 0} x^n - \frac{1}{12}\sum_{n\geq 0}(-1)^n \left(\frac{x}{3}\right)^n$

$$f(x) = \sum_{n\geq 0} \frac{1}{4}\left(3 - \frac{(-1)^n}{3^{n+1}}\right) x^n$$

$ln(4+x) = ?$ We have : $\left(ln(4+x)\right)' = \frac{1}{4+x} = \frac{1}{4}\frac{1}{1+\frac{x}{4}} = \frac{1}{4}\sum_{n\geq 0}(-1)^n \left(\frac{x}{4}\right)^n = \sum_{n\geq 0} (-1)^n \frac{x^n}{4^{n+1}}$

By integration we obtint : $ln(4+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n\,4^n} x^n$