I. Numerical series

1) Definitions :

Consider the sequence $(u_n)_n$, $u_n \in IK$ $(IK = \mathbb{R}$ ou $\mathbb{C})$.
And (S_n) the sequence defined by : $S_n = u_0 + u_1 + u_2 + \cdots + u_n$
We call numérical serie the pair (S_n, u_n) formed by the sequences (u_n) and (S_n).
We call u_n the general term of the serie (S_n, u_n) .

$S_n = \sum_{k=0}^{k=n} u_k$ Called the partial sum of the serie (S_n, u_n) .

We say that serie (S_n, u_n) is convergent if and only if the sequence $(S_n)_n$ is convergent.

We call $s = \lim_{n\to+\infty} S_n = \sum_{n=0}^{+\infty} u_n$ the Sum of serie(S_n, u_n) .

A numérical serie that is not convergent, is said divergent. ($\lim_{n\to+\infty} S_n = \infty$ or does not exist.)

Examples :

1) $s = \sum_{n=0}^{+\infty} u_n$, $u_n = \frac{1}{(n+1)(n+2)}$

$$u_n = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$S_n = \sum_{k=0}^{n} u_k = \sum_{k=0}^{n} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$S_n = 1 - \frac{1}{n+2}$ And $s = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(1 - \frac{1}{n+2}\right) = 1$

.

Then : $\sum_{n=0}^{+\infty} \frac{1}{(n+1)(n+2)}$ is convergent, and its sum $s = 1$.

2) $\sum_{n=0}^{+\infty} ln\left(1+\frac{1}{n}\right)$, $u_n = ln\left(1+\frac{1}{n}\right) = ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$

$$S_n = \sum_{k=0}^{n} u_k = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + \cdots + (\ln(n) - \ln(n-1)) + (\ln(n+1) - \ln(n))$$

$S_n = \ln(n+1) - \ln(1) = \ln(n+1)$, $s = \lim_{n\to+\infty} S_n = +\infty$ Then : $\sum_{n=0}^{+\infty} ln\left(1+\frac{1}{n}\right)$ diverges.

2) Necessary condition of convergence :

$$\sum_{n=0}^{+\infty} u_n \text{ converges } \Rightarrow \lim_{n\to+\infty} u_n = 0$$

Remark :

$$\lim_{n\to+\infty} u_n = 0 \nRightarrow \sum_{n=0}^{+\infty} u_n \text{ convergente}$$

Examples :

1) $\sum_{n=0}^{+\infty} \frac{n}{n+1}$, $u_n = \frac{n}{n+1} \xrightarrow{n\to+\infty} 1$ Then : $\sum_{n=0}^{+\infty} \frac{n}{n+1}$ diverges.

2) $\sum_{n=0}^{+\infty} ln\left(1+\frac{1}{n}\right)$, $u_n = ln\left(1+\frac{1}{n}\right) \xrightarrow{n\to+\infty} 0$ And $\sum_{n=0}^{+\infty} ln\left(1+\frac{1}{n}\right)$ diverges.

(See the example 2)

3) Particular series :

1. Riemann' serie : $\sum_{n\geq 0} \frac{1}{n^\alpha}$ Converges iff $\alpha > 1$.
2. Harmonic serie : $\sum_{n\geq 0} \frac{1}{n}$ Diverges (Riemann α=1).
3. Geometric serie : $\sum_{n\geq 0} r^\alpha$ Converges iff $|r|<1$.
4. Serie of Bertrand : $\sum_{n\geq 0} \frac{1}{n^\alpha(\ln(n))^\beta}$ converges iff : $\begin{cases} \alpha > 1 \text{ et } \beta \in R \\ \beta > 1 \text{ et } \alpha = 1 \end{cases}$

4) Convergence criteria (Tests of convergence) :

Let the serie : $\sum_{n\geq 0} u_n$ with a general term u_n.

Theorem 01 : Test of Cauchy

$$\text{If } \lim_{n\to+\infty} \sqrt[n]{|u_n|} = l \text{ Exist, Then : } \begin{cases} \sum_{n\geq 0} u_n \text{ converges if } l < 1 \\ \sum_{n\geq 0} u_n \text{ diverges if } l > 1 \end{cases}$$

Examples :

$$1)\ \sum_{n\geq 0}\left(\frac{n+5}{2n+1}\right)^n \quad , \quad \sqrt[n]{|u_n|} = \frac{n+5}{2n+1} \xrightarrow{n\to+\infty} \frac{1}{2} < 1 \text{ Then } \sum_{n\geq 0}\left(\frac{n+5}{2n+1}\right)^n \text{ converges.}$$

$$2)\ \sum_{n\geq 0}\left(\frac{2}{3}\right)^{n^2} \quad , \quad \sqrt[n]{|u_n|} = \left(\frac{2}{3}\right)^n \xrightarrow{n\to+\infty} 0 < 1 \text{ Then } \sum_{n\geq 0}\left(\frac{2}{3}\right)^{n^2} \text{ converges.}$$

Theorem 02 : Test of D'Alembert

$$\text{If } \lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = l \text{ Exist, Then : } \begin{cases} \sum_{n\geq 0} u_n \text{ converges if } l < 1 \\ \sum_{n\geq 0} u_n \text{ diverges if } l > 1 \end{cases}$$

Examples :

$$1)\ \sum_{n\geq 0}\frac{2^n}{n!} \quad , \quad \left|\frac{u_{n+1}}{u_n}\right| = \frac{2}{n+1} \xrightarrow{n\to+\infty} 0 < 1 \text{ then } \sum_{n\geq 0}\frac{2^n}{n!} \text{ converges.}$$

$$2)\ \sum_{n\geq 0}\frac{(n+1)!}{n^2} \quad , \quad \left|\frac{u_{n+1}}{u_n}\right| = \left(\frac{n}{n+1}\right)^2 (n+2) \xrightarrow{n\to+\infty} +\infty \text{ then } \sum_{n\geq 0}\frac{2^n}{n!} \text{ diverges.}$$

Theorem 03 : Test of comparison

Let : $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ two series with positive terms .

$$\text{For } n \geq n_0 , \quad u_n \leq v_n \text{ Then : } \begin{cases} \sum_{n\geq 0} v_n \text{ Converges} \Rightarrow \sum_{n\geq 0} u_n \text{ Converges} \\ \sum_{n\geq 0} u_n \text{ Diverges} \Rightarrow \sum_{n\geq 0} v_n \text{ Diverges} \end{cases}$$

Examples :

1) $\sum_{n\geq 0}\frac{3^n}{1+5^n}$ $\quad u_n=\frac{3^n}{1+5^n}\leq\frac{3^n}{5^n}=\left(\frac{3}{5}\right)^n$ $\left(0<5^n\leq 1+5^n\Rightarrow\frac{1}{1+5^n}\leq\frac{1}{5^n}\right)$

The serie $\sum_{n\geq 0}\left(\frac{3}{5}\right)^n$ is geometric serie converges $\left(q=\frac{3}{5}<1\right)$.

Then : $\sum_{n\geq 0}\frac{3^n}{1+5^n}$ is convergent .

2) $\sum_{n\geq 2}\frac{1}{\sqrt{n^2-1}}$ $\quad u_n=\frac{1}{\sqrt{n^2-1}}\geq\frac{1}{n}$ $\quad\left(n\geq 2,\quad \sqrt{n^2-1}\leq\sqrt{n^2}=n\right)$.

$\sum_{n\geq 2}\frac{1}{n}$ Harmonic serie diverges, then $\sum_{n\geq 2}\frac{1}{\sqrt{n^2-1}}$ is divergent.

Theorem 04 : Test of equivalence

Let : $\sum_{n\geq 0}u_n$ and $\sum_{n\geq 0}v_n$ two series with positive terms.

If $\lim_{n\to+\infty}\frac{u_n}{v_n}=1$ Then : $\sum_{n\geq 0}u_n$ and $\sum_{n\geq 0}v_n$ have the same behavior.

Examples :

1) $\sum_{n\geq 0}\frac{2n^2+1}{n^5+4}$ $\quad u_n=\frac{2n^2+1}{n^5+4}\sim\frac{2n^2}{n^5}=\frac{2}{n^3}$

$\sum_{n\geq 0}\frac{2}{n^3}$ Riemann' serie converges $(\alpha=3>1)$. Then : $\sum_{n\geq 0}\frac{2n^2+1}{n^5+4}$ is convergent.

2) $\sum_{n\geq 0}\frac{7^n}{5^n+\ln(n)}$ $\quad u_n=\frac{7^n}{5^n+\ln(n)}\sim\frac{7^n}{5^n}=\left(\frac{7}{5}\right)^n$ $\quad\left(\lim_{n\to+\infty}\frac{\ln(n)}{5^n}=0\right)$.

$\sum_{n\geq 0}\left(\frac{7}{5}\right)^n$ Geometric serie diverges $\left(q=\frac{7}{5}>1\right)$. Then $\sum_{n\geq 0}\frac{7^n}{5^n+\ln(n)}$ Diverges.

Theorem 05 : Test of Abel

The serie $\sum_{n\geq 0}(u_n v_n)$ Converges if and only if $\begin{cases} (u_n) \text{ monotonic sequence and } \lim_{n\to+\infty} u_n = 0 \\ \exists M \geq 0 \quad \left|\sum_{k=0}^{n} v_k\right| \leq M \end{cases}$

Example :

$$\sum_{n\geq 0}\frac{(-1)^n}{2n^3+1} \qquad u_n = \frac{(-1)^n}{2n^3+1} = a_n b_n$$

$$\begin{cases} a_n = \dfrac{1}{2n^3+1} \text{ decreasing and } \lim_{n\to+\infty} a_n = 0 \\ \left|\sum_{k=0}^{n} b_k\right| = \left|\sum_{k=0}^{n}(-1)^n\right| = \left|\dfrac{1-(-1)^{n+1}}{2}\right| \leq \dfrac{1+|(-1)^{n+1}|}{2} = 1 \end{cases}$$

According to Abel test : $\sum_{n\geq 0}\frac{(-1)^n}{2n^3+1}$ is convergent.

5) Properties :

Let : $\sum_{n\geq 0} u_n , \sum_{n\geq 0} v_n$ numerical series, and α real number. We have :

$$\begin{cases} \text{If } \sum_{n\geq 0} u_n \text{ and } \sum_{n\geq 0} v_n \text{ convergs , Then } \sum_{n\geq 0} \alpha u_n \text{ and } \sum_{n\geq 0}(u_n+v_n) \text{ are convergents} \\ \text{If } \sum_{n\geq 0} u_n \text{ converges and } \sum_{n\geq 0} v_n \text{ diverges , then } \sum_{n\geq 0}(u_n+v_n) \text{ is divergent} \\ \text{If } \sum_{n\geq 0} u_n \text{diverges and } \sum_{n\geq 0} v_n \text{ diverges, we don't conclure} \end{cases}$$

Example :

Let : $\sum_{n\geq 0}\left(\frac{1}{5^n}+2^n\right)$ and $\sum_{n\geq 0}\left(\frac{1}{5^n}-2^n\right)$ diverges series.

Put : $u_n = \frac{1}{5^n}+2^n$ and $v_n = \frac{1}{5^n}-2^n$

$$\begin{cases} u_n + v_n = \dfrac{2}{5^n} \\ u_n - v_n = 2^{n+1} \end{cases} \Rightarrow \begin{cases} \sum_{n\geq 0}(u_n+v_n) = \sum_{n\geq 0}\dfrac{2}{5^n} \text{ is convergent (geometric serie } q = \dfrac{1}{5} < 1) \\ \sum_{n\geq 0}(u_n-v_n) = \sum_{n\geq 0} 2^{n+1} \text{ is divergent(geometric serie } q = 2 > 1) \end{cases}$$

6) Alternating serire :

We call alternating serie every serie given by the form :

$$\sum_{n\geq 0}(-1)^n u_n \quad \text{where } (u_n) \text{ have a constant sign.}$$

Theorem of Leibnitz :

If : $\begin{cases} \text{The sequence } (u_n) \text{ is monotonic} \\ \lim\limits_{n\to+\infty} u_n = 0 \end{cases}$ Then $\sum\limits_{n\geq 0}(-1)^n u_n$ is converges.

Examples :

1) $\sum\limits_{n\geq 1}\frac{(-1)^n}{\sqrt{n}}$ On a $\left(\frac{1}{\sqrt{n}}\right)$ decreasing serie. and $\frac{1}{\sqrt{n}} \xrightarrow{n\to+\infty} 0$.

Then $\sum\limits_{n\geq 1}\frac{(-1)^n}{\sqrt{n}}$ is convergent.

2) $\sum\limits_{n\geq 1}(-1)^n sin\left(\frac{1}{n}\right)$ we have $\left(sin\left(\frac{1}{n}\right)\right)' = \frac{-1}{n^2}cos\left(\frac{1}{n}\right) < 0, \left(sin\left(\frac{1}{n}\right)\right)$ decreasing,

and $sin\left(\frac{1}{n}\right) \xrightarrow{n\to+\infty} 0$. Then (Leibnitz), $\sum\limits_{n\geq 1}(-1)^n sin\left(\frac{1}{n}\right)$ is convergent.

7) Absolute convergence and conditional convergente :

Let : $\sum\limits_{n\geq 0} u_n$ a numerical serie.

a) The serie $\sum\limits_{n\geq 0} u_n$ is absolutely convergente iff the serie $\sum\limits_{n\geq 0}|u_n|$ is convergent.

b) The serie $\sum\limits_{n\geq 0} u_n$ is conditionaly-converges iff the serie $\sum\limits_{n\geq 0} u_n$ converges and the serie $\sum\limits_{n\geq 0}|u_n|$ diverges.

Proposition :

$$\sum_{n\geq 0}|u_n| \text{ converges} \Longrightarrow \sum_{n\geq 0} u_n \text{ converges.}$$

Examples :

1) $\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n}}$ converges (according to Leibnitz). $\sum_{n\geq 1}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n\geq 1}\frac{1}{\sqrt{n}}$ Riemann' serie,

$\left(\alpha = \frac{1}{2} < 1\right)$ diverges. Then $\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n}}$ conditionaly-converges.

2) $\sum_{n\geq 0}\frac{(-1)^n}{2n^3+n^2+3}$ $\quad u_n = \frac{(-1)^n}{2n^3+n^2+3}$, $|u_n| = \frac{1}{2n^3+n^2+3} \sim \frac{1}{2n^3}$

$\sum_{n\geq 1}\frac{1}{2n^3}$ Riemann' serie converges ($\alpha = 3 > 1$).

Then : $\sum_{n\geq 0}\frac{(-1)^n}{2n^3+n^2+3}$ is absolutely converges.

And $\sum_{n\geq 0}\frac{(-1)^n}{2n^3+n^2+3}$ is convergent.