Larbi Ben M'hidi University

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Physics 1

Mechanics of the material point

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Online courses

Mechanics of the material point

Chapter 1: Kinematics of a material point

- Movement characteristics
- Rectilinear motion
- Motion in a Plane
- Movement in space

Motion in a Plane (2D)

If the trajectory of a moving point lies within a plane, we can describe its position using either Cartesian coordinates $(x,\ y)$or polar coordinates (r, φ).

- ✓ Cartesian basis vectors $(\vec{i}, \vec{j})$ are fixed.
- ✓ Polar basis vectors $(\overrightarrow{U_r}, \overrightarrow{U_\varphi})$ depend on the position θ.

Study of movement in polar coordinates

Mobile position

The position of the mobile in Cartesian coordinates (x, y) is defined by :

$$\boxed{\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j}}$$

- Position vector in Cartesian coordinates (x, y)

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j}$$

- Position vector in Polar coordinates (r, φ)

$$\boxed{\overrightarrow{OM} = \vec{r} = r\,\overrightarrow{U_r}}$$

r: is the distance from the origin O to the point M.

r=f(t)

φ : is the angle between the position vector $\vec{r}$ and a the x-axis.

φ=g(t)

Unit vectors $\boxed{\overrightarrow{U_r} \perp \overrightarrow{U_\varphi}}$

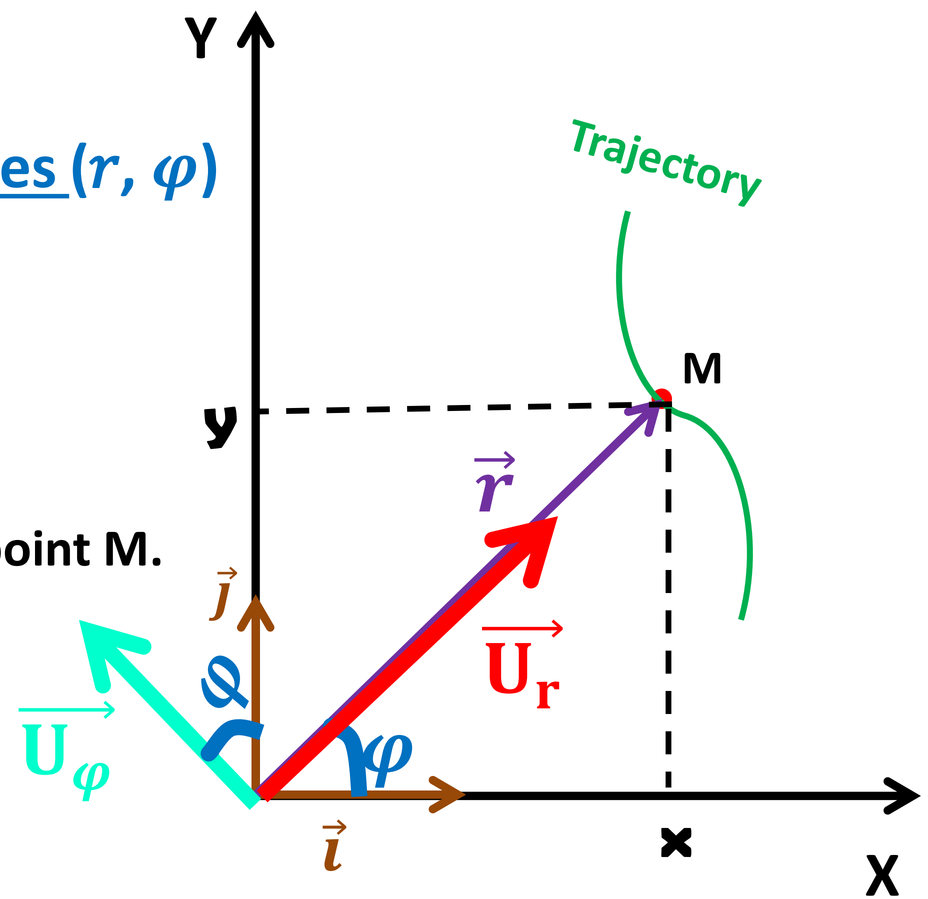


Figure 1: Position vector

Expression of Unit vectors $\overrightarrow{U_r}$ and $\overrightarrow{U_\theta}$

$$\overrightarrow{U_r} = \overrightarrow{U_{rx}} + \overrightarrow{U_{ry}} \qquad (*')$$

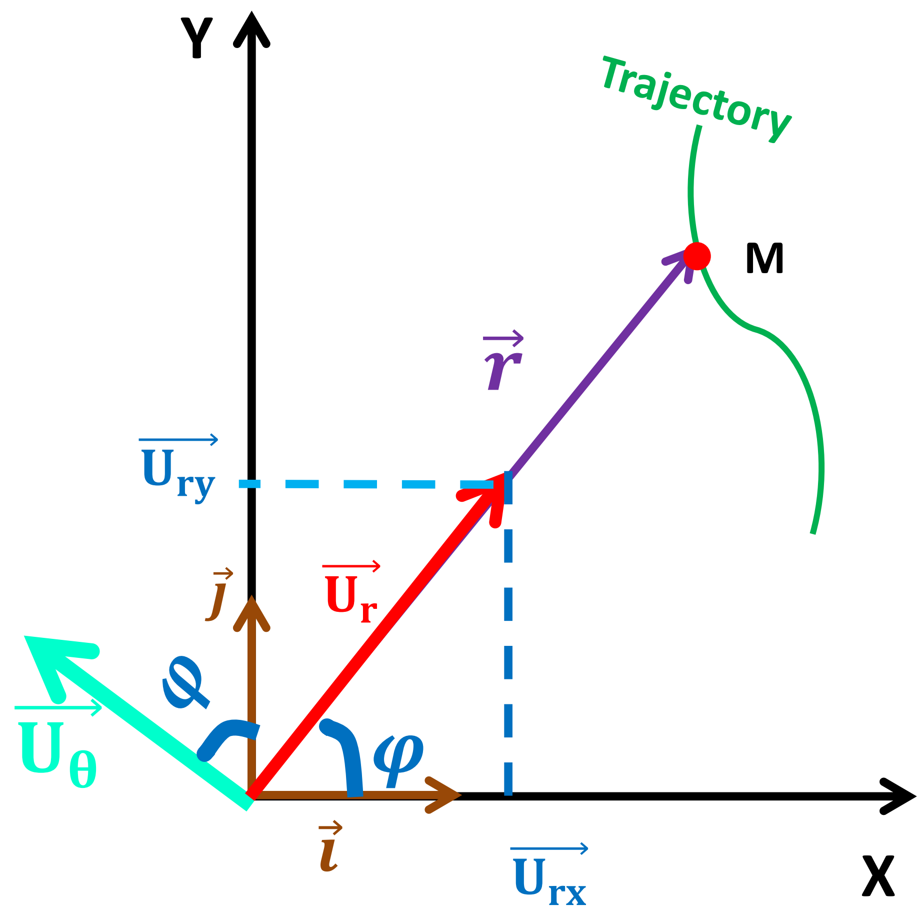
$$\begin{cases} \overrightarrow{U_r} = U_r\overrightarrow{U_r} & (1) \\ \overrightarrow{U_{rx}} = U_{rx}\,\vec{i} & (2) \\ \overrightarrow{U_{ry}} = U_{ry}\,\vec{j} & (3) \end{cases}$$

$(1), (2) and\ (3) \rightarrow (*') \Rightarrow$

$$U_r\overrightarrow{U_r} = U_{rx}\,\vec{i} + U_{ry}\,\vec{j} \quad (*'')$$

$$\begin{cases} cos\varphi = \dfrac{U_{rx}}{U_r} \Rightarrow U_{rx} = U_r\cos\varphi & (4) \\ \sin\varphi = \dfrac{U_{ry}}{U_r} \Rightarrow U_{ry} = U_r\sin\varphi & (5) \end{cases}$$

$(4), (5) \rightarrow (*'') \Rightarrow \cancel{U_r}\overrightarrow{U_r} = \cancel{U_r}cos\varphi\,\vec{i} + \cancel{U_r}sin\varphi\,\vec{j} \Rightarrow$



$$\overrightarrow{U_r} = cos\varphi\,\vec{i} + sin\varphi\,\vec{j}$$
$$\overrightarrow{U_\varphi} = -sin\varphi\,\vec{i} + cos\varphi\,\vec{j}$$
$(\blacksquare)$

$\overrightarrow{U_r}$ and $\overrightarrow{U_\varphi}$ *are variable*

$(\blacksquare) \rightarrow$ ★ $\Rightarrow$

$$\overrightarrow{OM} = \vec{r} = r\,\underbrace{(cos\varphi\,\vec{i} + sin\varphi\,\vec{j})}_{\overrightarrow{U_r}}$$

Position vector in Polar coordinates (r, φ)

Relation between Cartesian coordinates (x, y) and polar coordinates (r, φ) is:

❖ Polar→→→→ Cartesian

$$\cos\varphi = \frac{x}{r} \Rightarrow x = r\cos\varphi$$

$$\sin\varphi = \frac{y}{r} \Rightarrow y = r\sin\varphi$$

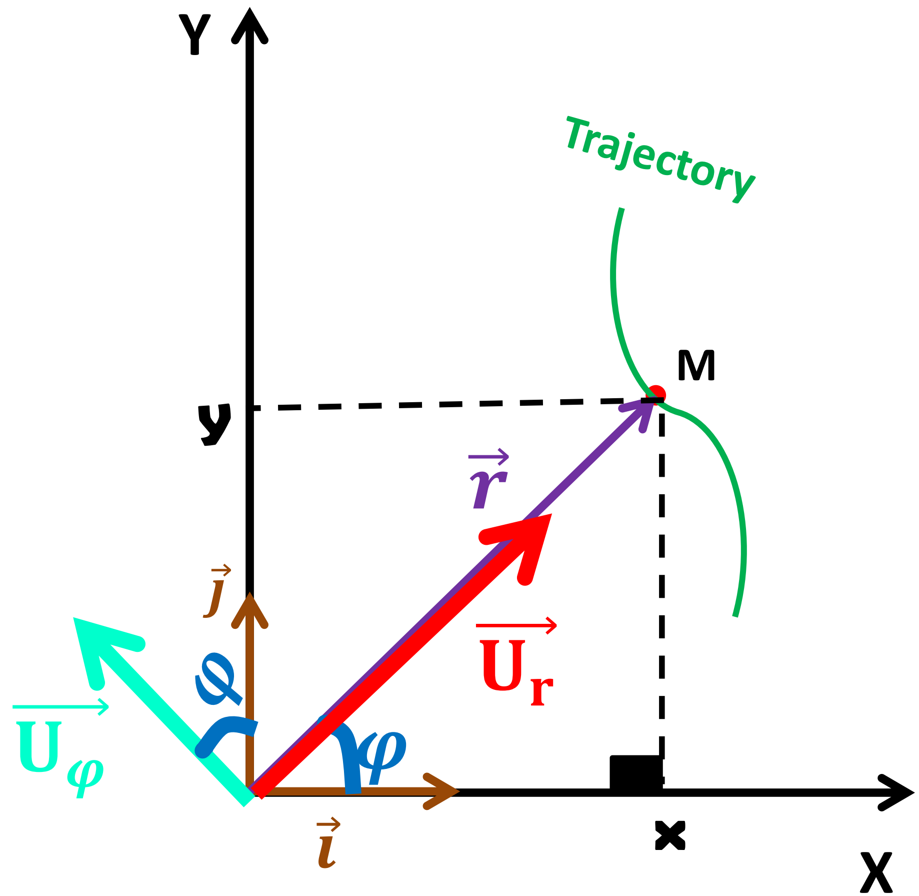
❖ Cartesian →→→→Polar

$$r = \sqrt{x^2 + y^2}$$

$$\varphi = arc\cos\frac{x}{r}$$

$$\varphi = arc\sin\frac{y}{r}$$

$$\varphi = arc\tan\frac{y}{x}$$



Relation between Cartesian basis $(\vec{i}, \vec{j})$ and polar basis $(\overrightarrow{U_r}, \overrightarrow{U_\varphi})$ is:

The polar unit vectors $(\overrightarrow{U_r}, \overrightarrow{U_\varphi})$ can be expressed in terms of the Cartesian unit vectors $(\vec{i}, \vec{j})$ as:

$$\overrightarrow{U_r} = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}$$
$$\overrightarrow{U_\varphi} = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$$

$\overrightarrow{U_r}$ **and** $\overrightarrow{U_\varphi}$ *are variable*

And conversely, The Cartesian unit vectors $(\vec{i}, \vec{j})$ can be expressed in terms of the polar unit vectors $(\overrightarrow{U_r}, \overrightarrow{U_\varphi})$ as:

$$\vec{i} = \cos\theta\,\overrightarrow{U_r} - \sin\theta\,\overrightarrow{U_\varphi}$$
$$\vec{j} = \sin\theta\,\overrightarrow{U_r} + \cos\theta\,\overrightarrow{U_\varphi}$$

Expression of velocity

- **In Cartesian coordinates:** $\boxed{\vec{v} = \dot{\vec{r}} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath}}$

- **In polar coordinates**

We have : $\overrightarrow{OM} = \vec{r} = r\,\overrightarrow{U_r}$ and $\overrightarrow{U_r} = cos\varphi\,\vec{i} + sin\varphi\,\vec{j}$ (6)

$$\Rightarrow \vec{v} = \dot{\vec{r}} = \frac{dr}{dt}\overrightarrow{U_r} + r\frac{d\overrightarrow{U_r}}{dt} \quad (7) \text{ with } \begin{cases} \dot{r} = \dfrac{dr}{dt} \\ \dot{\overrightarrow{U_r}} = \dfrac{d\overrightarrow{U_r}}{dt} \end{cases} \qquad \dot{\varphi} = \frac{d\varphi}{dt}$$

$$(6) \Rightarrow \frac{d\overrightarrow{U_r}}{dt} = -\frac{d\varphi}{dt}sin\varphi\,\vec{i} + \frac{d\varphi}{dt}cos\varphi\,\vec{j} \Rightarrow \frac{d\overrightarrow{U_r}}{dt} = \dot{\varphi}\,\underbrace{(-sin\varphi\,\vec{i} + cos\varphi\,\vec{j})}_{\overrightarrow{U_\varphi}}$$

$$\Rightarrow \boxed{\frac{d\overrightarrow{U_r}}{dt} = \dot{\varphi}\,\overrightarrow{U_\varphi}} \quad (8)$$

$$\frac{d\overrightarrow{U_r}}{dt} = \dot{\varphi}\,\overrightarrow{U_\varphi}$$

$$\frac{d\overrightarrow{U_\varphi}}{dt} = -\dot{\varphi}\,\overrightarrow{U_r}$$

$(8) \rightarrow (7) \Rightarrow$ $$\boxed{\vec{\mathrm{v}} = \dot{\vec{\mathrm{r}}} = \underbrace{\dot{r}\overrightarrow{\mathrm{U}_{\mathrm{r}}}}_{v_r} + \underbrace{r\dot{\varphi}\,\overrightarrow{U_{\varphi}}}_{v_{\varphi}}}$$

In polar coordinates

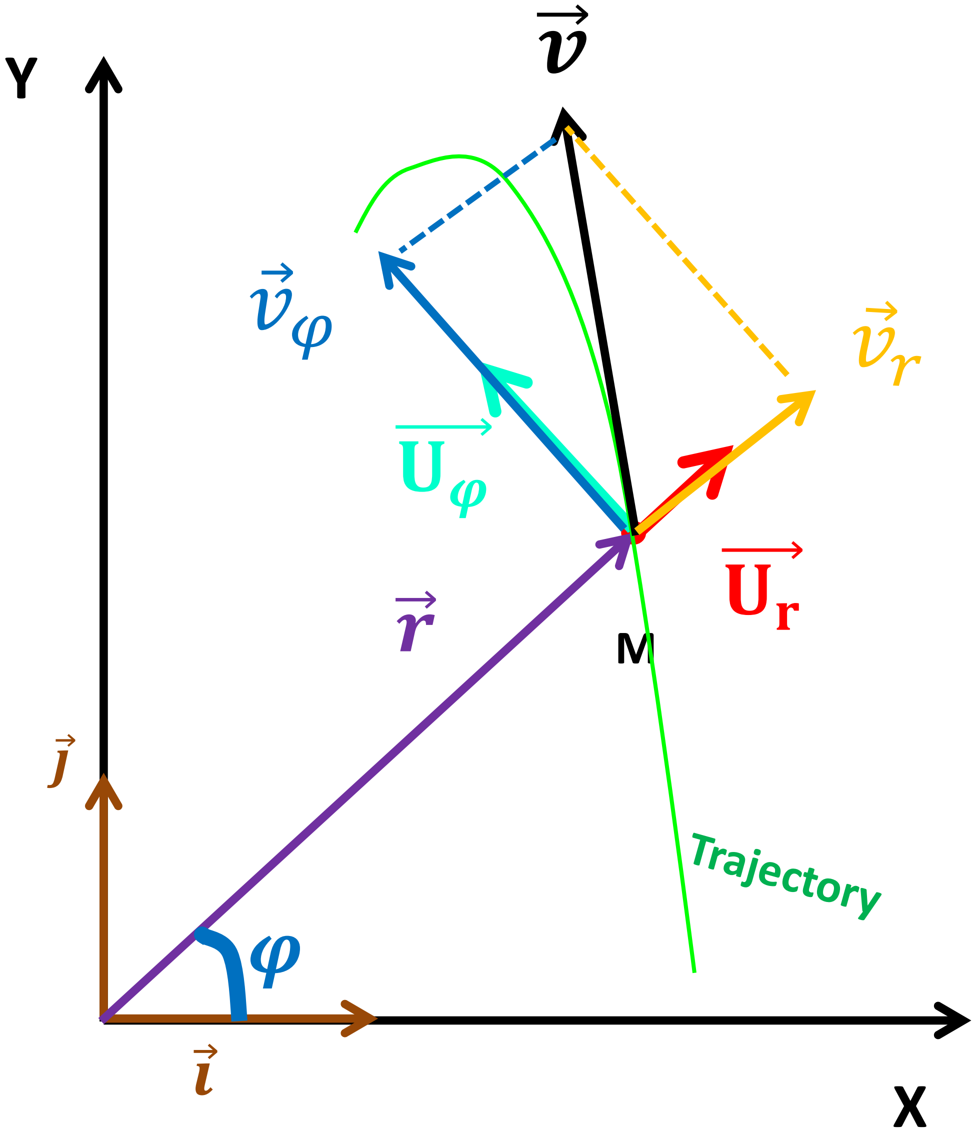
$$\Rightarrow \boxed{\vec{\mathrm{v}} = \overrightarrow{v_{\mathrm{r}}} + \overrightarrow{v_{\varphi}}}$$

➢ Magnitude of the Velocity Vector

$$v = \sqrt{{v_r}^2 + {v_{\varphi}}^2} \Rightarrow \boxed{v = \sqrt{\dot{r}^2 + (r\dot{\varphi})^2}}$$

The radial velocity (v_r) is defined as the rate of change of the radial distance "r" with respect to time. Mathematically, it is expressed as: $v_r = \dot{r}$. This component indicates how quickly the particle is moving towards or away from the origin.

The transverse velocity (v_{φ}), also known as tangential velocity, represents the component of velocity that is perpendicular to the radial direction. It is given by: $v_{\varphi} = r\dot{\varphi}$. This component reflects how fast the particle is moving along its circular path at a given radius.



Y
$\vec{v}$
$\vec{v}_\varphi$
$\vec{v}_r$
$\overrightarrow{U_\varphi}$
$\overrightarrow{U_r}$
$\vec{r}$
M
$\vec{j}$
Trajectory
φ
$\vec{i}$
X

Expression of Acceleration

- In Cartesian coordinates: $\boxed{\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath}}$

- <u>In polar coordinates</u>

We have : $\vec{v} = \dot{r}\overrightarrow{U_r} + r\dot{\varphi}\,\overrightarrow{U_\varphi}$

$$\Rightarrow \vec{a} = \dot{\vec{v}} = \underbrace{\frac{d\dot{r}}{dt}}_{\ddot{r}}\overrightarrow{U_r} + \dot{r}\underbrace{\frac{d\overrightarrow{U_r}}{dt}}_{\dot{\varphi}\overrightarrow{U_\varphi}} + \underbrace{\frac{dr}{dt}}_{\dot{r}}\dot{\varphi}\overrightarrow{U_\varphi} + r\underbrace{\frac{d\dot{\varphi}}{dt}}_{\ddot{\varphi}}\overrightarrow{U_\varphi} + r\dot{\varphi}\underbrace{\frac{d\overrightarrow{U_\varphi}}{d}}_{-\dot{\varphi}\overrightarrow{U_r}}$$

$$\Rightarrow \vec{a} = \ddot{r}\overrightarrow{U_r} + \dot{r}\dot{\varphi}\overrightarrow{U_\varphi} + \dot{r}\dot{\varphi}\overrightarrow{U_\varphi} + r\ddot{\varphi}\overrightarrow{U_\varphi} + r\dot{\varphi}\left(-\dot{\varphi}\overrightarrow{U_r}\right)$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\overrightarrow{U_r} + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\overrightarrow{U_\varphi}}$$

$$\Rightarrow \quad \boxed{\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\overrightarrow{U_r} + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\overrightarrow{U_\varphi}}$$

$$\underbrace{\phantom{(\ddot{r} - r\dot{\varphi}^2)}}_{a_r} \qquad \underbrace{\phantom{(2\dot{r}\dot{\varphi} + r\ddot{\varphi})}}_{a_\theta}$$

$$\boxed{\vec{a} = \overrightarrow{a_r} + \overrightarrow{a_\varphi}}$$

Acceleration Vector in Polar Coordinates

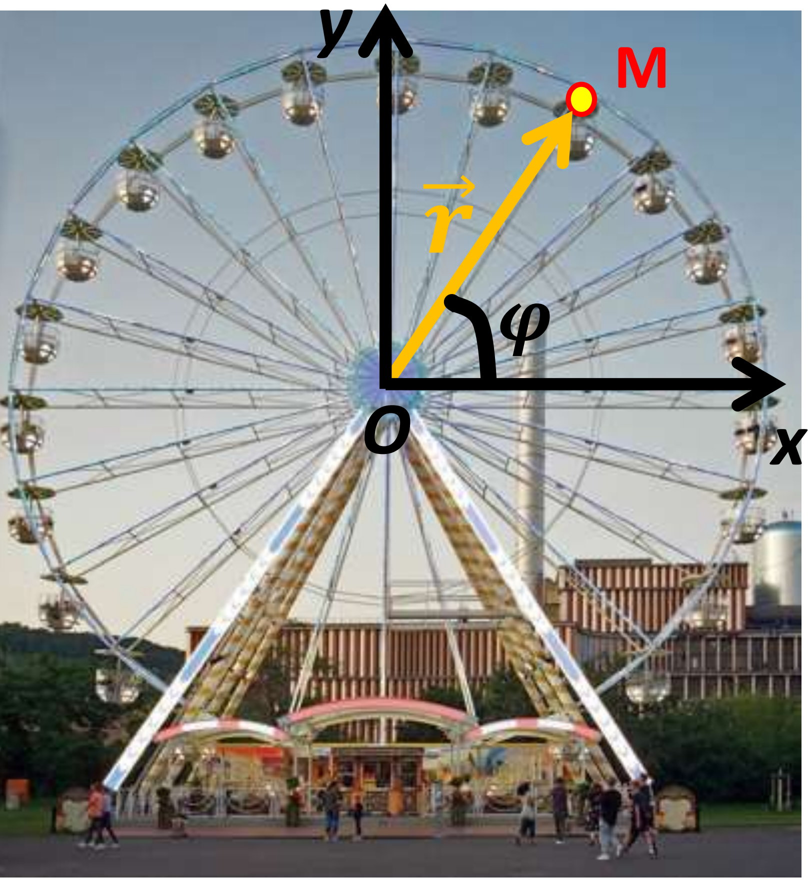
a_r :The radial component of acceleration.

a_ω :The tangential component of acceleration.

- **Magnitude of the acceleration Vector**

$$a = \sqrt{{a_r}^2 + {a_\varphi}^2} \quad \Rightarrow \quad \boxed{a = \sqrt{(\ddot{r} - r\dot{\varphi}^2)^2 + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})^2}}$$

Special Cases 1: Circular Motion (Motion in a Plane (2D))



We have $\boxed{r=R=C}$ $\qquad \begin{cases} \dot{r} = 0 \\ \ddot{r} = 0 \end{cases}$

The velocity vector is therefore:

$$\vec{v} = \underbrace{\cancel{\dot{r}\overrightarrow{U_r}}}_{0} + r\dot{\varphi}\,\overrightarrow{U_\varphi}$$

$$\Rightarrow \boxed{\vec{v} = R\dot{\varphi}\,\overrightarrow{U_\varphi}}$$

The velocity vector in Circular Motion

And the expression for the acceleration vector is:

$$\vec{a} = (\underbrace{\cancel{\ddot{r}}}_{0} - r\dot{\varphi}^2)\overrightarrow{U_r} + (\underbrace{\cancel{2\dot{r}\dot{\varphi}}}_{0} + r\ddot{\varphi})\overrightarrow{U_\varphi}$$

Acceleration vector in Circular Motion

$$\Rightarrow \boxed{\vec{a} = -\underbrace{R\dot{\varphi}^2}_{a_N}\overrightarrow{U_r} + \underbrace{R\ddot{\varphi}}_{a_T}\overrightarrow{U_\varphi}}$$

a_N : The normal acceleration always points towards the center of the circular path, which means it is directed radially inward ($a_N = a_r = R\dot{\varphi}^2$). It is responsible for changing the direction of the velocity vector without altering its magnitude.

a_T:Tangential acceleration is the component of acceleration that acts along the direction of motion, affecting the speed (magnitude) of the velocity vector. It measures how quickly an object's speed changes as it moves along its path. It can either be in the same direction as the velocity (increasing speed) or opposite to it (decreasing speed) ($a_\theta = a_T = R\ddot{\varphi}$).

Another special case, uniform circular motion

the uniform circular motion is characterized by the constant velocity (constant in module V=C) while its direction continuously changes.

We have $\vec{V} = R\dot{\varphi}\,\overrightarrow{U_{\varphi}}$ and $\boxed{\dot{\varphi} = \omega} \Rightarrow \boxed{\ddot{\varphi} = \frac{d\omega}{dt} = 0}$

module $\Rightarrow \boxed{V = R\dot{\varphi} = R\omega = C} \Rightarrow \boxed{Angular\ velocity\ \omega\ ({}^{rad}/_{s}) = C}$

For acceleration :

We have $\vec{a} = -R\underbrace{\dot{\varphi}}_{\omega}{}^{2}\,\overrightarrow{U_r} + \cancel{R\ddot{\varphi}}^{\,0}\,\overrightarrow{U_{\varphi}}$

$\Rightarrow \boxed{\vec{a} = \underbrace{-R\omega^{2}}_{a_N}\overrightarrow{U_r}}$ $\Rightarrow$ module $\boxed{a = R\omega^{2}}$

$$\Rightarrow \boxed{a = a_r = a_N = R\omega^2 = \frac{V^2}{R}}$$

Normal and Tangential Components of Velocity and Acceleration in the Frenet Frame

In the study of motion along a curved path, the Frenet frame provides a systematic way to analyze the motion by breaking down velocity and acceleration into tangential and normal components.

The Frenet frame consists of two orthogonal unit vectors:

- **Tangent vector (T): in the direction of motion.**
- **Normal vector (N): in towards the center of curvature.**

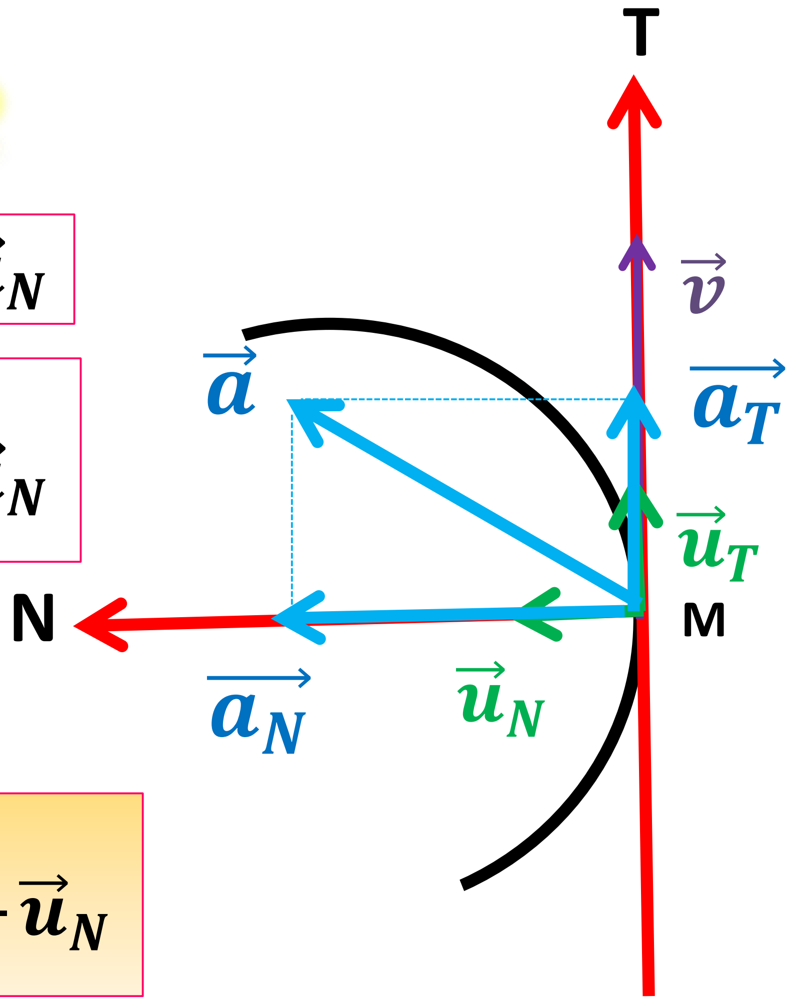
$$\boxed{\vec{v} = v\vec{u}_T}$$

$$\vec{a} = \overrightarrow{a_T} + \overrightarrow{a_N} \quad \Rightarrow$$

$$\vec{a} = a_T\vec{u}_T + a_N\vec{u}_N$$

$$\vec{a} = \frac{dv}{dt}\vec{u}_T + \frac{v^2}{R}\vec{u}_N$$

$$\Rightarrow \quad \boxed{\vec{a} = \dot{v}\vec{u}_T + \frac{v^2}{R}\vec{u}_N}$$



Velocity and acceleration in the Frenet frame