Larbi Ben M'hidi University

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Physics 1

Mechanics of the material point

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Online courses

Mechanics of the material point

Chapter 1: Kinematics of a material point

- Movement characteristics
- Rectilinear motion
- Plane motion
- Movement in space
- Relative motion

Chapter 1: Kinematics of a material point

Characteristics of movement

- Position of the mobile
- Time equations
- The velocity vector
- The acceleration vector

Chapter 1: kinematic

MOVEMENT CHARACTERISTICS

1- Introduction

- **Kinematics** consists of analyzing the movement of "points" without worrying about the causes of this movement. We will therefore not talk here about forces or Newton's laws. This chapter is purely mathematical.

- **The material point** is any material body whose dimensions are theoretically zero and practically negligible compared to the distance traveled.

❖ Kinematic magnitude

- Position $\overrightarrow{OM}$
- Velocity $\vec{v}$
- Acceleration $\vec{a}$

Anyone wishing to study a movement must first impose a reference in relation to which the movement is analyzed.

This study of movement takes one of two forms:

- ✓ **Vector** ➡ **Using the vectors: position $\overrightarrow{OM}$, Velocity $\vec{v}$ and acceleration $\vec{a}$.**
- ✓ **Algebraic** ➡ **By defining the equation of motion along a given trajectory.**

2. Mobile position

The position of a material point M at time t is represented in a orthonormal reference system R $(O, \vec{i}, \vec{j}, \vec{k})$ by a position vector $\overrightarrow{OM}$ (See figure 1).

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

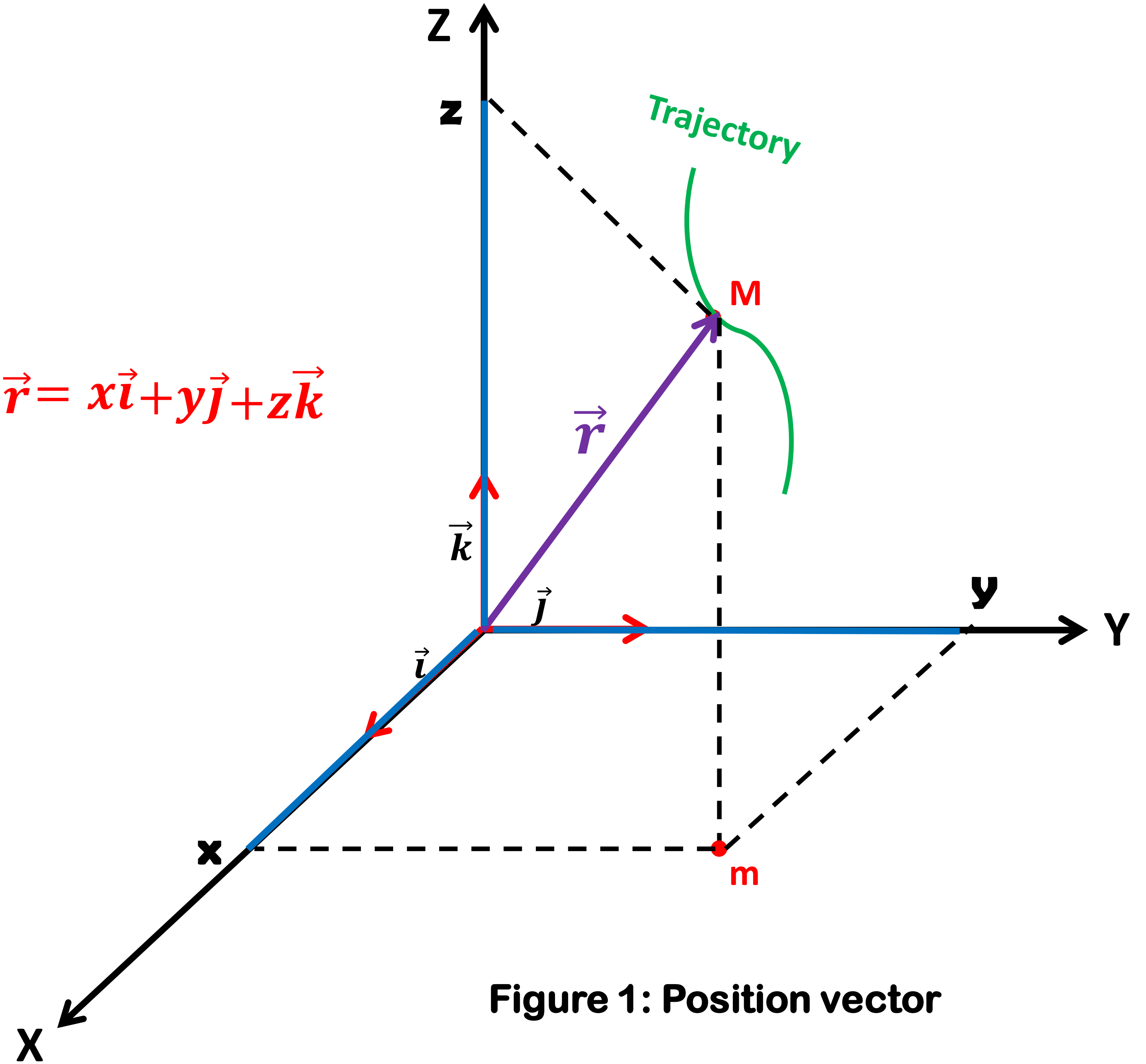


Figure 1: Position vector

The formula that expresses the position vector in Cartesian coordinates.

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \overrightarrow{OM}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Where (x, y, z) (Cartesian coordinates) are the components of the vector $\overrightarrow{OM}$ in the basis $(\vec{i}, \vec{j}, \vec{k})$

3. Time equations ⟷ Equations of Motion

- **Material Point at Rest:** A material point is considered at rest in a chosen reference system if : Its Cartesian coordinates (x, y, z) are independent of time.
- **Material Point in Motion :** A material point is in movement in a given reference system if : At least one of its Cartesian coordinates varies as a function of time. These coordinates can be noted by:

$$\boxed{x(t), y(t), z(t)}$$

These functions are called the time equations of motion. They can be expressed in the form:

$$\boxed{x = f(t), y = g(t), z = h(t)}$$

4- Trajectory

The trajectory is the set of positions occupied by the mobile during its movement during successive instants.

The trajectory equation is obtained by eliminating the time between the two time equations.

Example 1

We consider a material point M moving in space R $\left(O,\vec{i},\vec{j},\vec{k}\right)$. The time equations of this movement describe the coordinates x(t), y(t) and z(t) of the point M as a function of time t . These equations are:

$$x = t+1 \quad ; \quad z=0 \quad ; \quad y = t^2+1$$

1/ Find the Cartesian equation of the trajectory, what is its form?

2/ Write the expression of the position vector at time t= 1s.

Solution x= t+1 (1) ; z=0 (2) ; y= t^2+1 (3)

1/ We take t from the equation x , which we replace by y:

$(1) \rightarrow$ x=t+1 ⇨ **t=x-1** $(*)$

(*) → (3) ⇨ $y = (x-1)^2 + 1$ ⇨ $\boxed{y = x^2 - 2x + 2}$

so the trajectory described by point M is a parabolic trajectories.

Notes

The general equation of a parabola is:

$$y = ax^2 + bx + c$$

2. Expression of the position vector $\overrightarrow{OM}$ at time t= 1s

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$ with x= t+1 ; z=0 ; y= t^2+1

$$\overrightarrow{OM} = \vec{r} = \underbrace{(t+1)}_{x}\vec{i} + \underbrace{(t^2+1)}_{y}\vec{j} + \underbrace{0}_{z}\vec{k}$$ With t=1s

$$\Rightarrow \quad \overrightarrow{OM} = \vec{r} = (1+1)\vec{i} + (1^2+1)\vec{j} + 0\vec{k}$$

$$\overrightarrow{OM} = \vec{r} = 2\vec{i} + 2\vec{j} + 0\vec{k} \quad \Rightarrow \quad \overrightarrow{OM} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Example 2

The time equations of the material point M moving in space R $\left(O, \vec{i}, \vec{j}, \vec{k}\right)$ are:

$$x = t \quad ; \qquad y = 0 \quad ; \qquad z = -2t^2 + 2t$$

-what is the trajectory followed ?

Solution

$$x = t \quad (1) \; ; \qquad y = 0 \quad (2) ; \qquad z = -2t^2 + 2t \quad (3)$$

- We take t from the equation x , which we replace by z:

$(1) \rightarrow$ t=x ⇒ (3) ⇔ $z = -2x^2 + 2x$

so the trajectory described by point M is a parabolic trajectories.

5-The velocity vector

Velocity is considered to be the distance traveled per unit of time.

5.1. Average velocity vector ($\vec{v}_{avg}$)

The average velocity of a body that moves between two points M and M' is defined as the ratio between the displacement vector and the time interval in which the displacement takes place.

$$\vec{v}_{avg} = \frac{\overrightarrow{MM'}}{t_2 - t_1}$$

$$\overrightarrow{v_{avg}} // \overrightarrow{MM'}$$

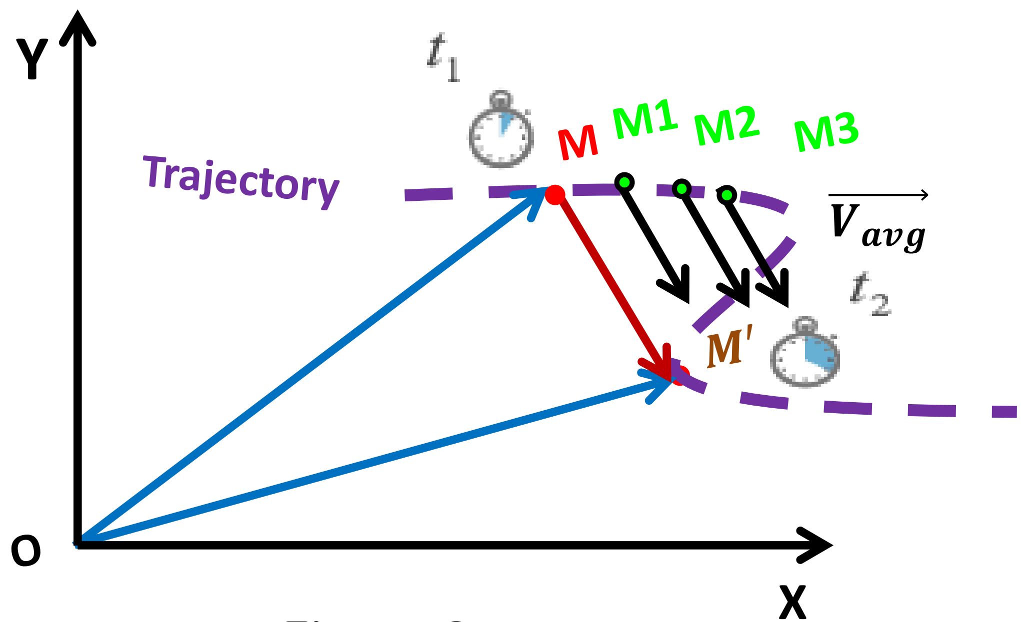


Figure 2

The average velocity vector is defined as follows;

$$\vec{v}_{avg} = \frac{\overrightarrow{MM'}}{\Delta t} \quad \textbf{With } \Delta t = t_2 - t_1$$

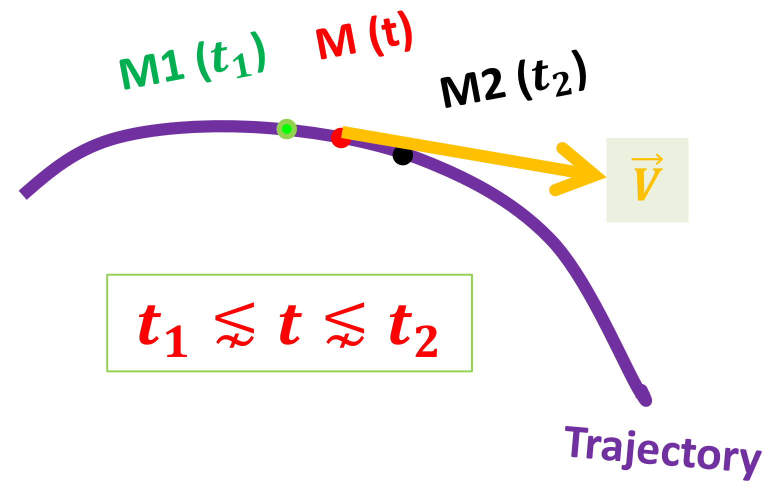
where:

- $\overrightarrow{v_{avg}}$**: Average velocity vector in the time studied.**
- $\overrightarrow{MM'}$ **: Displacement vector in the time studied.**
- $t_1,\ t_2$ **: Time in which the body is in the initial M and final M' points respectively**

$$\overrightarrow{MM'} = \overrightarrow{OM'}(t_2) - \overrightarrow{OM}(t_1) = \Delta\overrightarrow{OM} \Rightarrow \vec{v}_{avg} = \frac{\Delta\overrightarrow{OM}}{\Delta t}$$

5.2. Instantaneous velocity vector ($\vec{v}$)

$$\vec{v}(t) = \frac{\overrightarrow{M_1M_2}}{\Delta t}$$ With $\Delta t = t_2 - t_1 \lll$



$$t_1 \lnsim t \lnsim t_2$$

$$\vec{v}(t) = \frac{\Delta\overrightarrow{OM}}{\Delta t}$$ With $\Delta t = t_2 - t_1 \lll$

we replace Δt by dt

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt}$$

The instantaneous velocity vector at time t is the derivative of the position vector $\overrightarrow{OM}$ with respect to time.

In the Cartesian coordinates for example, we deduce the expression of the instantaneous velocity vector from the expression of the position vector by deriving:

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

$$\Downarrow$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\vec{v} = \begin{pmatrix} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{z} = v_z \end{pmatrix}$$

Where

$$\dot{x} = \frac{dx}{dt};\ \dot{y} = \frac{dy}{dt};\ \dot{z} = \frac{dz}{dt}$$

The instantaneous velocity vector $\vec{v}$ is carried by the tangent to the trajectory at point M; it is always oriented in the direction of movement.

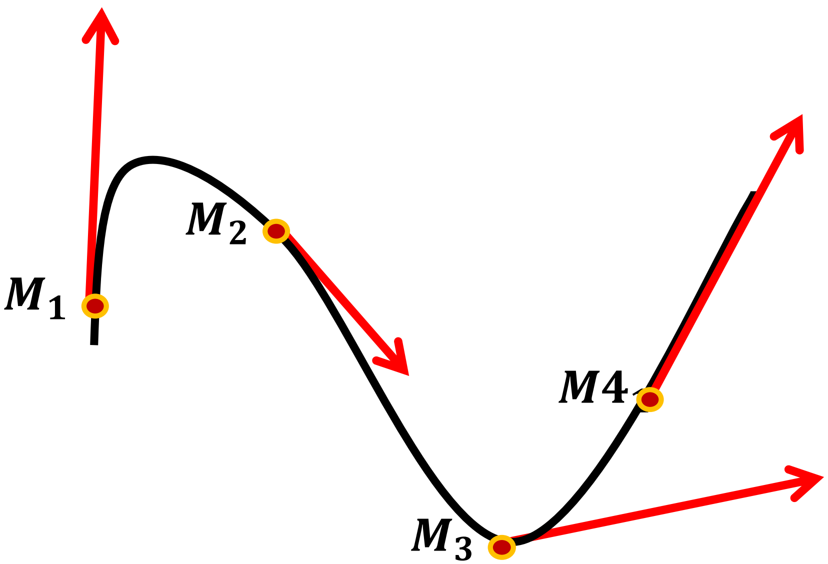


Figure 3: instantaneous velocity vector

Magnitude of the instantaneous velocity vector

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

The SI unit of velocity is (m/s)

Application

We consider a mobile with position vector $\overrightarrow{OM} = 3t\vec{\imath} - 2t^2\vec{\jmath}$

1. Calculate $\vec{V}(t)$.
2. Deduce its norm (magnitude) at date "t".
3. Calculate velocity at date t=2s.

Solution

1. $$\overrightarrow{OM} = \overbrace{3t\vec{\imath}}^{x} - \overbrace{2t^2\vec{\jmath}}^{y} \Rightarrow \vec{V}(t) = \frac{d\overrightarrow{OM}}{dt} \Rightarrow \begin{cases} \checkmark\ \vec{v} = \dfrac{dx}{dt}\vec{\imath} + \dfrac{dy}{dt}\vec{\jmath} \\ \checkmark\ \vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} \end{cases}$$

$$\boxed{\vec{V}(t) = \underbrace{3\vec{\imath}}_{v_x = \dot{x}} - \underbrace{4t\vec{\jmath}}_{v_y = \dot{y}}}$$

NB:
$$(f^m)' = mf^{m-1}$$

2. $$\boxed{V = \sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow V = \sqrt{3^2 + (-4t)^2} \Rightarrow \boxed{V = \sqrt{9 + 16t^2}}$$

3. $V = \sqrt{9 + 16t^2}$ **With t=2s.**

$$\Rightarrow V = \sqrt{9 + 16(2)^2} \Rightarrow \boxed{V = \sqrt{73} = 8.54\ ({}^{m}/_{s})}$$

6. The acceleration vector

We consider acceleration to be the change in velocity per unit time. The SI unit of acceleration is (m/s²)

6.1. Average acceleration vector ($\vec{a}_{avg}$)

Considering two different times t_1 and t_2 corresponding to the position vectors $\overrightarrow{OM}$ and $\overrightarrow{OM'}$ and the instantaneous velocity vectors $\vec{v}$ and $\vec{v'}$ (see Figure 4)

$$\vec{a}_{avg} = \frac{\vec{v'} - \vec{v}}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

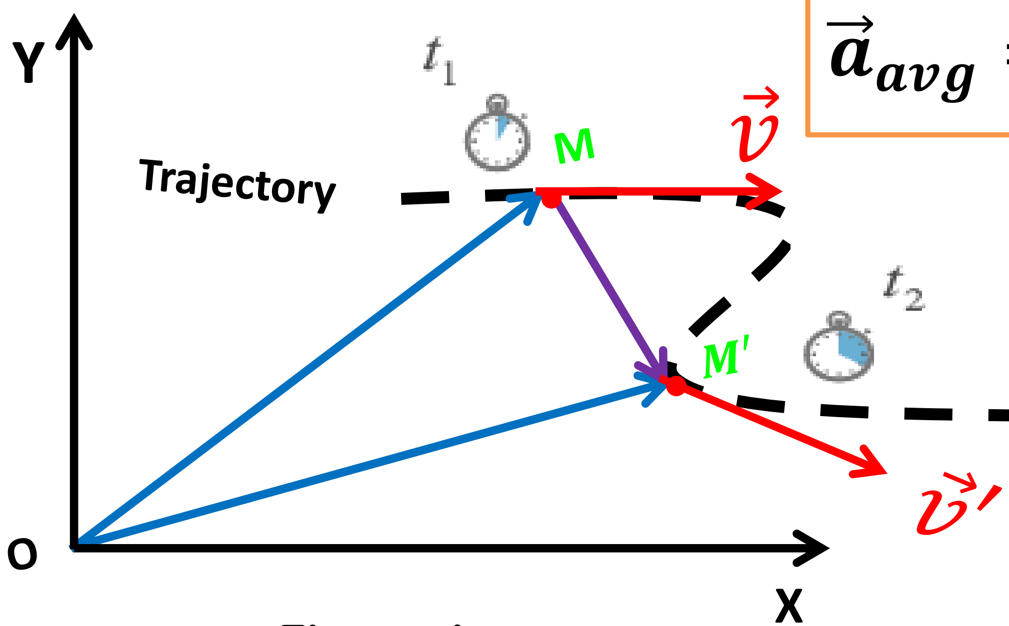


Figure 4

the average acceleration vector is defined by the expression:

$$\boxed{\vec{a}_{avg} = \frac{\vec{v\prime} - \vec{v}}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}}$$

6.2. Instantaneous acceleration vector ($\vec{a}$)

The instantaneous acceleration vector of a motion is defined as the derivative of the instantaneous velocity vector with respect to time.

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}}$$

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} \Rightarrow \boxed{\vec{a} = \overbrace{\ddot{x}}^{a_x}\vec{i} + \overbrace{\ddot{y}}^{a_y}\vec{j} + \overbrace{\ddot{z}}^{a_z}\vec{k}}$$

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \Rightarrow \boxed{\vec{a} = \underbrace{\frac{d^2x}{dt^2}}_{a_x}\vec{i} + \underbrace{\frac{d^2y}{dt^2}}_{a_y}\vec{j} + \underbrace{\frac{d^2z}{dt^2}}_{a_z}\vec{k}}$$

Magnitude of the instantaneous acceleration vector

the magnitude of the acceleration is

$$\boxed{a = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}$$

The acceleration vector is always directed towards the inside of the curvature of the trajectory.

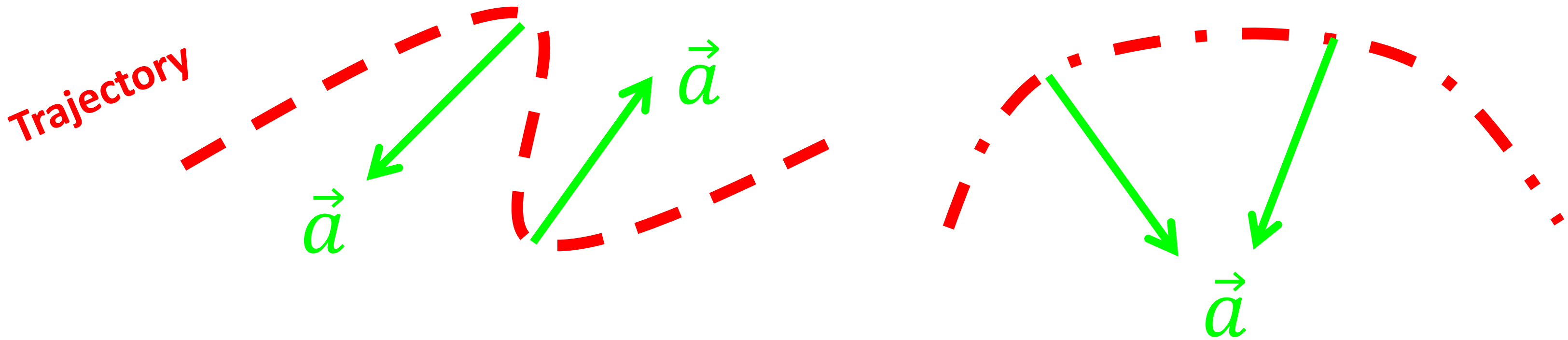


Figure 5: acceleration vector

Notes

- The movement is <u>accelerated</u> ($v\uparrow$) if ; $\vec{a} \cdot \vec{v} > 0$,
- The movement <u>decelerated</u> or <u>retarded</u> ($v\downarrow$) if ; $\vec{a} \cdot \vec{v} < 0$.

Example 3

The position vector is $\overrightarrow{OM}\begin{pmatrix} 3t \\ 2t^3+1 \\ t^2-3 \end{pmatrix}$, deduce the instantaneous velocity vector and the acceleration vector, then calculate the magnitude of each of them.

Solution

$$\boxed{\overrightarrow{OM} = 3t\vec{\imath} + (2t^3+1)\vec{\jmath} + (t^2-3)\vec{k}}$$

$$\boxed{\vec{v} = 3\vec{\imath} + 6t^2\vec{\jmath} + 2t\vec{k}} \Rightarrow v = \sqrt{3^2 + (6t^2)^2 + (2t)^2} \Rightarrow \boxed{v = \sqrt{9 + 36t^4 + 4t^2}}$$

$$\boxed{\vec{a} = 0\vec{\imath} + 12t\vec{\jmath} + 2\vec{k}} \Rightarrow a = \sqrt{0^2 + (12t)^2 + 2^2} \Rightarrow \boxed{a = \sqrt{144t^2 + 4}}$$