Chapter 3

Series

3.1 Numerical Series

3.1.1 Generality

Definition: *We call a numerical series a sequence with the general term u_n:*

$$\sum_{k=0}^{+\infty} u_k = u_0 + u_1 + \ldots + u_n + \cdots$$

The partial sum of the first n terms is denoted as:

$$s_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^{n} u_k$$

Definition 3.1 *We say that the numerical series $\sum_{k=0}^{+\infty} u_k$ is convergent if and only if its sequence of partial sums (S_n) converges, meaning:*

$$\lim_{n\to\infty} S_n \text{ exists and is finite. Otherwise, we say that it is divergent.}$$

Example 3.1 *Determine the nature of the series with the general term $u_n = \frac{1}{n(n+1)}, n \geqslant 1$:*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

thus, $s_n = u_1 + \ldots + u_n = 1 - \frac{1}{n+1}$, and $\lim_{n\to\infty} u_n = 1$, so the numerical serie is convergent.

Proposition 3.1 *If the series* $\sum_{n=0} u_n$ *converges, then* $\lim_{n\to\infty} u_n = 0$. *Conversely:*

$$\lim_{n\to\infty} u_n \neq 0 \Rightarrow \sum_{n\geq 0} u_n \textit{ diverges}$$

Example 3.2 *Consider* $u_n = \left(1 + \frac{1}{n}\right)^n$*:*

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1 + \frac{1}{n})^n = e \neq 0 \Rightarrow$$
$$\sum_{n>0} \left(1 + \frac{1}{n}\right)^n \textit{ diverges.}$$

3.1.2 Series with Positive Terms

A) Comparison Test: *Let* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *be two numerical series with positive terms such that:*

$$\forall n \in \mathbb{N},\ u_n \leq v_n$$

- *If* $\sum_{n\geq 0} v_n$ *converges, then* $\sum_{n\geq 0} u_n$ *also converges, and* $\sum_{n\geq 0} u_n \leq \sum_{n\geq 0} v_n$.
- *If* $\sum_{n\geq 0} u_n$ *diverges, then* $\sum_{n\geq 0} v_n$ *also diverges.*

Remark 3.1 *In general, one compares a series with a reference series:*

The sum of geometric series $\sum_{n\geq 0} u_n$, *with* $u_0 \neq 0$ *so we have.*

The geometric series $\sum_{n\geq 0} u_n$ *converges if and only if* $|q| < 1$. *In this case:*

$$\sum_{n\geq 0} u_n = \sum_{n\geq 0} u_0 q^n = \frac{u_0}{1-q}.$$

Riemann Series

The Riemann series is a series with general term U_n *of the form* $\frac{1}{n^\alpha}$, *with* $\alpha \in \mathbb{R}_+$. *We have:*

$$\sum_{n\geq 1} \frac{1}{n^\alpha}.$$

The Riemann series converges if $\alpha > 1$ *and diverges if* $\alpha \leq 1$.

Bertrand Series

The Bertrand series is of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}(\ln n)^{b}},$$

which converges for $\alpha > 1$ or if $\alpha = 1$ and $\beta > 1$.

B) Equivalence Criterion

Theorem 3.1 (Equivalence Criterion) *Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two numerical series with positive terms such that $u_n \approx v_n$, i.e., u_n is asymptotically equivalent to v_n if*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1.$$

Then, both series have the same nature (they both converge or both diverge).

Example 3.3 *Consider the series $\sum_{n=0} \frac{1}{n^2+3}$. We have:*

$$\frac{1}{n^2+3} \approx \frac{1}{n^2}.$$

Since

$$\lim_{n\to\infty} \frac{n^2}{n^2-3} = 1,$$

we conclude that $\sum_{n\geq 0} \frac{1}{n^2+3}$ converges.

C) d'Alembert's Criterion

Theorem 3.2 (d'Alembert's Ratio Test) *Let $\sum u_n$ be a series with positive terms. Define the limit:*

$$l = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|.$$

Then:

1. *The series converges if $l < 1$.*
2. *The series diverges if $l > 1$.*
3. *If $l = 1$, the test is inconclusive.*

Example 3.4 *Consider the series* $\sum_{n>0} \frac{\sqrt{3n}}{2^{n+1}}$. *We compute:*

$$\lim_{n\to\infty} \frac{u_{n-1}}{u_n} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{1}{2} = \frac{1}{2} < 1.$$

Therefore, $\sum_{n>0} \frac{\sqrt{3n}}{2^{n+1}}$ *converges.*

D) Cauchy's Root Test

Theorem 3.3 (Cauchy's Root Test) *Let* $\sum u_n$ *be a series with positive and non-zero terms such that*

$$\lim \sqrt[n]{u_n} = l.$$

Then:

1. *If* $l < 1$, *the series* $\sum u_n$ *converges.*
2. *If* $l > 1$, *the series* $\sum u_n$ *diverges.*
3. *If* $l = 1$, *the test is inconclusive.*

Example 3.5 *Consider the series* $\sum_{n=0} \left(\frac{1}{n}\right)^n$. *We have* $\sqrt[n]{u_n} = \frac{1}{n}$ *and* $u_n > 0$.

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, $\sum \left(\frac{1}{n}\right)^n$ *converges.*

3.1.3 Series of Arbitrary Sign

Definition 3.2 *A series is said to be of arbitrary sign if among its terms, there exist both positive and negative terms.*

Example 3.6 *An alternating series has a general term of the form*

$$u_n = (-1)^n v_n, \quad \text{where } u_n \text{ alternates in sign.}$$

Such a series is of arbitrary sign.

Remark 3.2 *Alternating series of the form* $\sum(-1)^n v_n$ *or* $\sum(-1)^{n+1} v_n$, *where* $v_n > 0$, *converge if the sequence* (v_n) *is decreasing and if* $\lim_{n\to\infty} v_n = 0$.

Example 3.7 *Consider the series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}.$$

We have $v_n = \frac{1}{n} > 0$, v_n *is decreasing, and* $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n} = 0$. *Thus, the series* $\sum_{n>0} \frac{(-1)^n}{n}$ *converges.*

3.1.4 Absolute Convergence

Definition 3.3 *A numerical series* $\sum_{n\geq 0} u_n$ *with arbitrary terms is said to be absolutely convergent if the series* $\sum_{n\geq 0} |u_n|$ *is convergent.*

Example 3.8 *The geometric series* $\sum q^n$ *converges absolutely if* $|q| < 1$, *as* $|q^n| = |q|^n$.

Example 3.9 *The series* $\sum \frac{(-1)^n}{n^2}$ *converges absolutely since*

$$\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2},$$

which corresponds to a Riemann series with $\alpha = 2 > 1$.

Proposition 3.2 *An absolutely convergent series is also convergent.*

Theorem 3.4 (Abel's Criterion) *Let* $\sum u_n$ *be a series with arbitrary terms such that* $u_n = a_n b_n$, *where:*

- *The sequence* b_n *is decreasing and converges to 0.*
- a_n *is a bounded sequence, i.e.,* $\exists M > 0$ *such that* $|\sum_{k=0}^{n} b_k| \leq M$ *for all* $n \in \mathbb{N}$.

Then, the series $\sum u_n$ *converges.*

Example 3.10 *Consider the series* $\sum \frac{(-1)^n}{n}$, *where* $a_n = \frac{1}{n}$ *(with* $a_n > 0$*) tends toward* 0, *and* $b_n = (-1)^n$. *Thus,* $\sum \frac{(-1)^n}{n}$ *converges.*

3.2 Sequence and Series of Functions

3.2.1 Sequence of Functions

Definition 3.4 *A sequence of functions is any sequence where the general term is a function depending on a parameter, denoted by* $(f_n)_n$.

Types of Convergence

Definition 3.5 (Pointwise Convergence) *Let $(f_n)_n$ be a sequence of functions defined on an interval $I \subset \mathbb{R}$. We say that $(f_n)_n$ converges pointwise to a function f on I if, for each $x \in I$, the sequence $(f_n(x))_n$ converges to a finite limit, denoted by $f(x)$.*

Definition 3.6 (Absolute Convergence) *We say that $(f_n)_n$ converges absolutely if the sequence formed by the absolute values of its terms is convergent.*

Definition 3.7 (Uniform Convergence) *We say that $(f_n)_n$ converges uniformly to f on I if*

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

Theorem 3.5 *Any uniformly convergent sequence is pointwise convergent; however, the converse is false.*

Properties of a Sequence of Functions

Let $(f_n)_n : I \to \mathbb{R}$ be a sequence of functions.

1. *If $(f_n)_n$ converges uniformly to f on I, then f is continuous on I.*
2. *If $(f_n)_n$ converges uniformly to f on I and each f_n is continuous, then f is also continuous.*
3. *If $(f_n)_n$ converges uniformly to f and each f_n is differentiable, then the sequence of derivatives $(f'_n)_{n\in\mathbb{N}}$ converges uniformly to f'.*
4. *Let $(f_n)_n$ be a sequence of Riemann-integrable functions that converges uniformly to a function f on $[a, b]$. Then,*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx.$$

Example 3.11 *Consider the sequence of functions $f_n(x) = x^n$ for $x \in \mathbb{R}$.*

1. *Simple Convergence:*

- *For* $|x| < 1$*: When* $|x| < 1$*, as* $n \to \infty$*, we have*

$$f_n(x) = x^n \to 0.$$

Therefore, the sequence (f_n) *converges pointwise to the function* $f(x) = 0$ *on the interval* $|x| < 1$*.*

- *For* $|x| = 1$*: When* $x = 1$*,* $f_n(1) = 1^n = 1$ *for all* n*, so the sequence does not converge to 0 but to 1. When* $x = -1$*, the sequence oscillates between 1 and -1 depending on whether* n *is even or odd, so it does not converge.*

- *For* $|x| > 1$*: When* $|x| > 1$*,* $f_n(x) = x^n$ *grows without bound as* $n \to \infty$*, so the sequence diverges for* $|x| > 1$*.*

In summary, the sequence $f_n(x) = x^n$ *converges pointwise to*

$$f(x) = \begin{cases} 0 & \text{if } |x| < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

2. *Uniform Convergence: To determine uniform convergence, we need to examine whether* $f_n(x) \to 0$ *uniformly on* $|x| < 1$*. For uniform convergence on* $|x| < 1$*, given* $\epsilon > 0$*, we need to find* N *such that for all* $n \geq N$ *and* $x \in (-1, 1)$*,*

$$|f_n(x) - 0| = |x^n| < \epsilon.$$

Since $|x| < 1$*, we can choose* N *large enough such that* $|x|^N < \epsilon$ *for all* $x \in (-1, 1)$*. Thus,* $f_n(x) \to 0$ *uniformly on* $|x| < 1$*.*
In conclusion: - The sequence (f_n) *converges pointwise to* $f(x) = 0$ *on* $|x| < 1$*. - The sequence* (f_n) *converges uniformly to* $f(x) = 0$ *on* $|x| < 1$*.*

3.2.2 Series of Functions

Definition 3.8 *Let* $(f_n)_n$ *be a sequence of functions defined on* $I \subset \mathbb{R}$*. We call the infinite sum of these terms a series of functions, denoted by*

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots$$

Definition 3.9 *Let $(f_n)_n$ be a sequence of functions defined on $I \subset \mathbb{R}$. We say that the series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on I if the sequence of partial sums (S_n) converges pointwise on I.*

Definition 3.10 *We say that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely on I if the series $\sum_{n=1}^{\infty} |f_n(x)|$ converges pointwise on I.*

Definition 3.11 *We say that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I if the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ converges uniformly on I.*

The domain of convergence of a series of functions is the set of points where it converges absolutely.

Example 3.12 *Calculate the domain of convergence of the series*

$$\frac{1}{1+x^2} + \frac{1}{1+x^4} + \frac{1}{1+x^6} + \cdots$$

To find the domain of convergence, we analyze the general term $f_n(x) = \frac{1}{1+x^{2n}}$ of the series.

1. For $|x| < 1$: We have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+x^{2n}} = 1,$$

since $\lim_{n\to\infty} f_n(x) \neq 0$, the series diverges.

2. For $|x| = 1$: The series also diverges, as seen by:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2}.$$

3. For $|x| > 1$: We have

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{1+x^{2n}} \leq \sum_{n=1}^{\infty} \frac{1}{x^{2n}},$$

which is a convergent geometric series.

Therefore, the domain of convergence of the series of functions is $|x| > 1$.