# Chapter 3

# Binary Relations on a Set

**Definition 3.0.1**
A binary relation is defined as any proposition between two objects, which can be either true or false. We denote it as $x\mathcal{R}y$ and read it as "$x$ is related to $y$".

**Example 3.0.2**

1. The inequality "$\leq$" is a binary relation on $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{R}$.
2. Parallelism "$\|$" and orthogonality "$\perp$" are binary relations on the set of lines in a plane or space.
3. The inclusion "$\subset$" is a binary relation on $P(E)$ or on $E$. The relation "greater than" on the set of students is a binary relation.

**Definition 3.0.3**
Given a binary relation $\mathcal{R}$ between elements of a non-empty set $E$, we say that :

1. $\mathcal{R}$ is Reflexive
$$\Leftrightarrow \forall x \in E, \quad x\mathcal{R}x.$$
2. $\mathcal{R}$ is Transitive
$$\Leftrightarrow \forall x; y; z \in E, \quad (x\mathcal{R}y) \wedge (y\mathcal{R}z) \Rightarrow (x\mathcal{R}z).$$
3. $\mathcal{R}$ is Symmetric
$$\Leftrightarrow \forall x; y \in E, \quad (x\mathcal{R}y) \Rightarrow (y\mathcal{R}x).$$
4. $\mathcal{R}$ is Anti-Symmetric
$$\Leftrightarrow \forall x; y \in E, \quad (x\mathcal{R}y) \wedge (y\mathcal{R}x) \ \Rightarrow \ x = y.$$

## 3.1 Equivalence Relation

**Definition 3.1.1**
We say that a binary relation $\mathcal{R}$ on a set $E$ is an equivalence relation if it is Reflexive, Symmetric and Transitive.

Let $\mathcal{R}$ be an equivalence relation on a set $E$.

**Definition 3.1.2**

1. Two elements $x$ and $y \in E$ are *equivalent* if $x\mathcal{R}y$.
2. The equivalence class of an element $x \in E$, denoted as $\dot{x}$, is the set :
$$\dot{x} = \{y \in E, \ x\mathcal{R}y\}.$$

3. $x$ is called a representative of the equivalence class $\dot{x}$.
4. The quotient set of $E$ by the equivalence relation $\mathcal{R}$, denoted as $E/_{\mathcal{R}}$, is the set of equivalence classes of all elements of $E$.
5. $\dot{x} = \dot{y} \Leftrightarrow x\mathcal{R}y$.

## Example 3.1.3

In $\mathbb{R}$, we define the relation $\mathcal{R}$ by :

$$\forall x, y \in \mathbb{R}; x\mathcal{R}y \Leftrightarrow x^2 - 1 = y^2 - 1.$$

Show that $\mathcal{R}$ is an equivalence relation and give the quotient set $\mathbb{R}/_{\mathcal{R}}$.

1. $\mathcal{R}$ is an equivalence relation.
   (a) $\mathcal{R}$ is Reflexive, because
   $$\forall x \in \mathbb{R}, \quad x^2 - 1 = x^2 - 1 \Rightarrow x\mathcal{R}x.$$
   (b) $\mathcal{R}$ is Symmetric, because
   $$\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \Rightarrow x^2 - 1 = y^2 - 1.$$
   $$\Rightarrow y^2 - 1 = x^2 - 1.$$
   Since equality is symmetric
   $$\Rightarrow y\mathcal{R}x.$$
   (c) $\mathcal{R}$ is Transitive, because
   $$\forall x; y; z \in \mathbb{R}, \quad (x\mathcal{R}y) \wedge (y\mathcal{R}z) \Rightarrow (x^2 - 1 = y^2 - 1).$$
   $$y^2 - 1 = z^2 - 1.$$
   $$\Rightarrow (x^2 - 1 = z^2 - 1),$$
   because equality is Transitive.
   $$\Rightarrow x\mathcal{R}z.$$
   From **i)**, **ii)**, and **iii)**, we conclude that $\mathcal{R}$ is an equivalence relation.
2. Determine the quotient set $\mathbb{R}/_{\mathcal{R}}$.
   Let $x \in \mathbb{R}$. Then,
   $$\forall y \in \mathbb{R}, \quad x\mathcal{R}y \Leftrightarrow x^2 - 1 = y^2 - 1,$$
   $$\Leftrightarrow x^2 - y^2 = 0,$$
   $$\Leftrightarrow (x - y)(x + y) = 0,$$
   $$\Leftrightarrow (y = x) \vee (y = -x).$$
   Thus,
   $$\dot{x} = \{-x, x\}.$$
   Hence,
   $$\mathbb{R}/_{\mathcal{R}} = \{\{-x, x\}, \ x \in \mathbb{R}\}.$$

## Proposition 3.1.4

Let $\mathcal{R}$ be an equivalence relation on a set $E$. Then, the equivalence classes form a partition of $E$, meaning that every equivalence class is non-empty, the union of equivalence classes is equal to $E$, two equivalence classes are either disjoint or identical,

$$\forall x, y \in E; \dot{x} \cap \dot{y} = \phi, \quad \text{or} \quad \dot{x} = \dot{y}.$$

## 3.2 Order Relation

**Definition 3.2.1**

An order relation on a set $E$ is any binary relation that is Reflexive, Anti-Symmetric, and Transitive. We say that $(E;\mathcal{R})$ is an ordered set (by $\mathcal{R}$) an order relation is often denoted as $\leq$.

**Example 3.2.2**

The relation "$\subset$" is an order relation on $P(E)$.

1. "$\subset$" is Reflexive, because $\forall A \in P(E)$, we have $A \subset A$.
2. "$\subset$" is Transitive, because
$$\forall A, B, C \in P(E), \quad (A \subset B) \wedge (B \subset C) \Rightarrow A \subset C.$$
3. "$\subset$" is Anti-Symmetric, because
$$\forall A, B \in P(E), \quad (A \subset B) \wedge (B \subset A) \Rightarrow A = B.$$

From (1), (2), and (3) we deduce that "$\subset$" is an order relation on $E$.

**Example 3.2.3**

1. The relations "$\leq$", "$\geq$" are order relations on $\mathbb{R}$.
2. The relation "/" (divides) is an order relation on $\mathbb{N}$.

### 3.2.1 Total or Partial Order

**Definition 3.2.4**

Let $E$ be a set equipped with a relation of order $\mathcal{X}$. We say that $\mathcal{R}$ is a total order relation if
$$\forall x,\ y \in E, \quad x\mathcal{R}y \ or \ \ y\mathcal{X}x.$$
Otherwise, we say that the order is partial, meaning that $\exists x, y \in E$ ; $x$ is not related to $y$, and $y$ is not related to $x$.

**Example 3.2.5**

1. The relations "$\geq$", "$\leq$" are total order relations on $\mathbb{N}$ ; $\mathbb{Z}$ ; $\mathbb{Q}$ ; $\mathbb{R}$.
2. The relation "/" (divides) is a partial order relation on $\mathbb{N}$.
3. The relation "$\subset$" is a partial order relation on $P(E)$.

### 3.2.2 Least Element, Greatest Element

**Definition 3.2.6**

Let $(E, \leq)$ be an ordered set and $A$ a subset of $E$.

1. We say that $m \in A$ is the least element of $A$ (or a minimum) if :
$$\forall x \in A, \quad m \leq x.$$
2. We say that $M \in A$ is the greatest element of $A$ (or a maximum) if :
$$\forall x \in A, \quad x \leq M.$$

**Proposition 3.2.7**

[Uniqueness of the minimum and maximum] Let $(E, \leq)$ be an ordered set and $A$ a subset of $E$.

1. If $A$ has a minimum, it is unique : we denote it $min\ A$.
2. If $A$ has a maximum, it is unique : we denote it $max\ A$.

**Example 3.2.8**

We define on $\mathbb{Z}$, the order relation $\mathcal{R}$ as follows :

$$\forall n, m \in \mathbb{Z}^*, n\mathcal{R}m \Leftrightarrow \exists k \in \mathbb{N}^* : m = kn.$$

1. The least element of $\mathbb{Z}^*$ by $\mathcal{R}$ is 1
2. $\mathbb{Z}$ does not have a greatest element. Let $A, B$ be subsets of $\mathbb{Z}^*$ such that, $A = \{2, -6, -10, -14, -18, -20\}$, $B = \{7, 6, 2, 3, -42\}$ Determine the least element (min), the greatest element (max) of $A$ ; $B$ by the relation $\mathcal{R}$ if they exist.
3. min $A = 2$.
4. max $A = \nexists$.
5. min $B = \nexists$.
6. max $B = -42$.

### 3.2.3 Upper Bound, Lower Bound

**Definition 3.2.9**

Let $(E, \leq)$ be an ordered set and $A$ a subset of $E$.

1. We say that $m \in E$ is a lower bound of $A$ if

$$\forall x \in A, \quad m \leq x.$$

   We say that $A$ is bounded below.
   We say that $M \in E$ is an upper bound of $A$ if

$$\forall x \in A, \quad x \leq M.$$

2. We say that $A$ is bounded above.
3. We say that $A$ is bounded if it is bounded below and above.

### 3.2.4 Infimum and Supremum

**Definition 3.2.10**

Let $(E, \leq)$ be an ordered set and $A$ a subset of $E$.

1. If the set of lower bounds of $A$ has a greatest element, we call it the infimum of $A$ and denote it $inf\ A$.
2. If the set of upper bounds of $A$ has a least element, we call it the supremum of $A$ and denote it $sup\ A$.

**Proposition 3.2.11**

Let $(E, \leq)$ be an ordered set and $A$ a subset of $E$

1. If $A$ has a minimum, then it has an infimum and $min\ A = inf\ A$.
2. If $A$ has a maximum, then it has a supremum and $max\ A = sup\ A$.

### Note 3.2.12

The least element of $A$ ($min\ A$), if it exists, is a lower bound of $A$, however, a lower bound of $A$ may not be the least element of $A$ since it may not necessarily be in $A$. Similarly for the greatest element ($max$ $A$) and the upper bound. This remark expresses that the reverse implication in Proposition 3.2.11 is not always true

### Example 3.2.13

Let $E = \{1, a, 2, 5, \gamma\}$, consider $(P(E), \subset)$ ordered, and let $A$ be a subset of $P(E)$.

$$A = \{\{a,\ 2\}, \{2, 5, \gamma\}, \{1, 2, \gamma\}, \{a, 2, 5\}\}.$$

Then,

1. The lower bounds of $A$ are : $\phi$ and $\{2\}$.
2. $infA$ =$\{2\}$.
3. $A$ does not have a least element, because $infA \notin A$.
4. The only upper bound of $A$ is : $E = \{1, a, 2, 5, \gamma\}$.
5. $supA = E$.
6. $A$ does not have a greatest element, because $supA \notin A$.