II- Partial Differential equations (PDE)

1) Definitions
2) PDE of first order
3) PDE of second order

1) Definitions and anotations :

Definition 01 :

We call a partial differential equation (EDP) every relation between an unknown function $u$ of two variables and its partial derivatives. Written :

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \dots\dots, \frac{\partial^n u}{\partial x^n}, \frac{\partial^n u}{\partial y^n}, \frac{\partial^n u}{\partial x^{n-1} \partial y} \dots\dots, \frac{\partial^n u}{\partial x \partial y^{n-1}}\right) = 0$$

Definition 02 :

We call order of the PDE, the order of the highest derivative occurring in the equation.

Definition 03 :

We say that EDP is linear if and only if it is linear with respect to the function $u$ And its partial derivatives.

Exemples :

$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2} + e^{-x}$ PDE of second order, linear, and non-homogeneous equation.

$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} = 0$ PDE of second order, linear, and homogeneous equation.

$\frac{\partial u}{\partial x} - 2u^2 + \sin(x) = 0$ PDE of first order, non-linear, and non-homogeneous equation.

$\frac{\partial^2 u}{\partial x \partial y} = 0$ PDE of second order, linear, and homogeneous equation.

$u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^3 u}{\partial t^3} = 0$ PDE of third order, non-linear, and homogeneous equation.

2) PDE of first order

$$(E): \ A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y) + D(u)$$

$$\text{Or } \ A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y,u)$$

Where $A, B, C$, and $D$ continuous functions on the domain $\Omega \subseteq \mathbb{R}^2$.

Methods of resolution :

a) Method of characteristic curves :

$$(E): \; A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y,u)$$

Theorem :

The solution of the equation $(E)$ is given by : $\varphi(C_1, C_2) = 0$ where $C_1$ and $C_2$ two independent integrals of the characteristic system : $\dfrac{dx}{A(x,y)} = \dfrac{dy}{B(x,y)} = \dfrac{du}{C(x,y,u)}$

Remark :

$\varphi(C_1, C_2) = 0 \;\Rightarrow\; C_1 = f(C_2)$ where $f$ is an arbitrary function .

Examples :

$(E_1): \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} = y \qquad A(x,y) = 1,\; B(x,y) = -1,\; C(x,y) = y$

The characteristic system : $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{du}{y} \Longrightarrow \begin{cases} dx = \dfrac{dy}{-1} \\ -dy = \dfrac{du}{y} \end{cases} \Rightarrow \begin{cases} C_1 = x + y \\ C_2 = u + \dfrac{1}{2}y^2 \end{cases}$

The general solution is : $\varphi\left(x+y, u - \dfrac{1}{2}y^2\right) = 0$ or $u(x,y) = -\dfrac{1}{2}y^2 + f(x+y)$with $f$ arbitrary.

$(E_2): x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2 \qquad A(x,y) = x,\; B(x,y) = y,\; C(x,y) = 2$

We have : $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{du}{2} \Longrightarrow \begin{cases} \dfrac{dx}{x} = \dfrac{dy}{y} \\ \dfrac{dx}{x} = \dfrac{du}{2} \end{cases} \Rightarrow \begin{cases} C_1 = \dfrac{y}{x} \\ C_2 = -2\,ln(x) + u \end{cases}$

The general solution is given by : $\varphi\left(\dfrac{y}{x}, -2\,ln(x) + u\right) = 0$ or $u(x,y) = 2ln(x) + f\left(\dfrac{y}{x}\right)$

with $f$ is an arbitrary function .

Remark : The arbitrary functions are determined by initial conditions.

Examples :

$$(E)\ :\begin{cases}\dfrac{\partial u}{\partial x}+x\dfrac{\partial u}{\partial y}=u\\ u(0,y)=\sin(y)\end{cases}\quad \text{The characteristic system :}$$

$$\frac{dx}{1}=\frac{dy}{x}=\frac{du}{u}\quad\Longrightarrow\begin{cases}xdx=dy\\ dx=\dfrac{du}{u}\end{cases}\Rightarrow\begin{cases}C_1=y-\dfrac{1}{2}x^2\\ C_2=-x+ln(u)\end{cases}$$

Then general solution is :

$$\varphi\left(y-\frac{1}{2}x^2,-x+ln(u)\right)=0 \text{ ou } u(x,y)=e^{x+f\left(y-\frac{1}{2}x^2\right)}.$$

We have $u(0,y)=sin(y)\ \Rightarrow f(y)=\ ln\,(\sin(y))$ . Then the particular solution of $(E)$ is :

$$u(x,y)=e^{x+ln\left(sin\left(y-\frac{1}{2}x^2\right)\right)}.$$

b) Separation of variables :

$(E): A\dfrac{\partial u}{\partial x}+B\dfrac{\partial u}{\partial y}=0$ With $A$ and $B$ functions of one variable.

Put : $u(x,y)=X(x)\,Y(y)$

$$\frac{\partial u}{\partial x}=X'(x)Y(y),\quad \frac{\partial u}{\partial y}=X(x)Y'(y)$$

Replace in$(E)$ : $AX'(x)Y(y)+BX(x)Y'(y)=0$

We obtain a following system of ODE: $A\frac{X\prime}{X}=-B\frac{Y\prime}{Y}=k$ (constant)

Integrate the two EDO, and replace $X,Y$ in $u$.

Examples :

$(E_1)$ : $y\dfrac{\partial u}{\partial x}+x^2\dfrac{\partial u}{\partial y}=0$ Put : $u(x,y)=X(x)\,Y(y)$

$$\begin{cases}\dfrac{\partial u}{\partial x}=X'(x)Y(y)\\ \dfrac{\partial u}{\partial y}=X(x)Y'(y)\end{cases}\Rightarrow\ yX'(x)Y(y)+x^2X(x)Y'(y)=0\Rightarrow\frac{-1}{x^2}\frac{X'}{X}=\frac{1}{y}\frac{Y'}{Y}=k\ (\text{cte})$$

$$\begin{cases} \dfrac{-1}{x^2}\dfrac{X'}{X} = k \\ \dfrac{1}{y}\dfrac{Y'}{Y} = k \end{cases} \Rightarrow \begin{cases} \dfrac{X'}{X} = -kx^2 \\ \dfrac{Y'}{Y} = ky \end{cases} \Rightarrow \begin{cases} ln|X| = \dfrac{-k}{3}x^3 + c_1 \\ ln|Y| = \dfrac{k}{2}y^2 + c_2 \end{cases} \Rightarrow \begin{cases} X = C_1 e^{\frac{-k}{3}x^3} \\ Y = C_2 e^{\frac{k}{2}y^2} \end{cases}$$

Then : $u(x,y) = X(x)\,Y(y) = C_1 e^{\frac{-k}{3}x^3} C_2 e^{\frac{k}{2}y^2} = C_1 C_2 e^{\frac{-k}{3}x^3 + \frac{k}{2}y^2} = C e^{\frac{-k}{3}x^3 + \frac{k}{2}y^2}$.

$(E_2)$ : $\dfrac{\partial u}{\partial x} + 2xy^2 \dfrac{\partial u}{\partial y} = 0$ Put : $u(x,y) = X(x)\,Y(y)$

$$\begin{cases} \dfrac{\partial u}{\partial x} = X'(x)Y(y) \\ \dfrac{\partial u}{\partial y} = X(x)Y'(y) \end{cases} \Rightarrow X'(x)Y(y) + 2xy^2 X(x)Y'(y) = 0 \Rightarrow \frac{1}{2x}\frac{X'}{X} = -y^2\frac{Y'}{Y} = k \text{ (cte)}$$

$$\begin{cases} \dfrac{1}{2x}\dfrac{X'}{X} = k \\ -y^2\dfrac{Y'}{Y} = k \end{cases} \Rightarrow \begin{cases} \dfrac{X'}{X} = 2xk \\ \dfrac{Y'}{Y} = -\dfrac{k}{y^2} \end{cases} \Rightarrow \begin{cases} ln|X| = kx^2 + c_1 \\ ln|Y| = \dfrac{k}{y} + c_2 \end{cases} \Rightarrow \begin{cases} X = C_1 e^{kx^2} \\ Y = C_2 e^{\frac{k}{y}} \end{cases}$$

Then : $u(x,y) = X(x)\,Y(y) = C_1 C_2 e^{kx^2} e^{\frac{k}{y}} = C_1 C_2 e^{kx^2 + \frac{k}{y}} = C e^{kx^2 + \frac{k}{y}}$.

c) Coordinates method :

$(E)$ : $a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y} = c$ with $a, b,$ and $c$ constants.

To solve this equation put the change of variables : $\begin{cases} s = ax + by \\ t = bx - ay \end{cases}$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x} = a\dfrac{\partial u}{\partial s} + b\dfrac{\partial u}{\partial t} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial y} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial y} = b\dfrac{\partial u}{\partial s} - a\dfrac{\partial u}{\partial t} \end{cases}$$ Replace in the equation $(E)$, we obtain :

$$a^2\frac{\partial u}{\partial s} + b^2\frac{\partial u}{\partial s} = c \Rightarrow \frac{\partial u}{\partial s} = \frac{c}{a^2+b^2} \Rightarrow u(x,y) = \frac{c}{a^2+b^2}s + f(t), \quad f \text{ Arbitrary.}$$

Then : $u(x,y) = \frac{c}{a^2+b^2}(ax+by) + f(bx-ay)$ .

Examples :

$$(E_1): \quad \frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = 2$$

Put : $\begin{cases} s = x-2y \\ t = -2x-y \end{cases}$

$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial s} - 2\frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -2\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \end{cases}$ Replce in the equation $(E_1)$, we obtain :

$5\frac{\partial u}{\partial s} = 2 \;\Rightarrow\; \frac{\partial u}{\partial s} = \frac{2}{5} \;\Rightarrow\; u(x,y) = \frac{2}{5}s + f(t)$, $f$ Arbitrary function.

Then : $u(x,y) = \frac{2}{5}(x-2y) + f(-2x-y)$ .

$(E_2): \begin{cases} 4\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = 0 \\ u(x,y) = y^3 \end{cases}$

Put : $\begin{cases} s = 4x-3y \\ t = -3x-4y \end{cases}$

$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = 4\frac{\partial u}{\partial s} - 3\frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -3\frac{\partial u}{\partial s} - 4\frac{\partial u}{\partial t} \end{cases}$ Replace in the equation $(E)$, we obtain :

$25\frac{\partial u}{\partial s} = 0 \;\Rightarrow\; \frac{\partial u}{\partial s} = 0 \;\Rightarrow\; u(x,y) = f(t)$, $f$ Arbitrary function.

Then the general solution is : $u(x,y) = f(-3x-4y)$ .

We have : $u(0,y) = y^3 \;\Rightarrow f(-4y) = y^3 \;\Rightarrow f(y) = \frac{-1}{64}y^3$

Then : $u(x,y) = \frac{-1}{64}(-3x-4y)^3 = \frac{1}{64}(3x+4y)^3$ .

3) Linear PDE of second order

The equation of the form :

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+D(x,y)\frac{\partial u}{\partial x}+F(x,y)\frac{\partial u}{\partial y}+G(x,y)u=H(x,y)$$

Is called a linear PDE of second order in $\Omega\subseteq\mathbb{R}^2$.

We can pose : $D(x,y)\frac{\partial u}{\partial x}+F(x,y)\frac{\partial u}{\partial y}+G(x,y)u-H(x,y)=F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$

The equation becomes : $A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}=F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$

Classification of PDE of second order

Put : $\Delta= B^2-4AC$ we distinguish three cases:

$$\begin{cases}\Delta>0 \text{ The equation is called hyperbolic}\\ \Delta=0 \text{ The equation is called Parabolic}\\ \Delta<0 \text{ The equation is called Elliptic}\end{cases}$$

Examples :

$\frac{\partial^2 u}{\partial x^2}-5\frac{\partial^2 u}{\partial x\partial y}=0$ $\quad\Delta=25>0$ then the equation is hyperbolic.

$\frac{\partial^2 u}{\partial x^2}+3x\frac{\partial u}{\partial y}=-e^x$ $\quad\Delta=0$ Then the PDE is parabolic.

$y^2\frac{\partial^2 u}{\partial x^2}+x^2\frac{\partial^2 u}{\partial y^2}+3u=0$ $\quad\Delta=-4x^2y^2$ Then the PDE is $\begin{cases}\text{Elliptic if } x\,,y\neq 0\\ \text{Parabolic if } x \text{ or } y=0\end{cases}$.

Methods of resolution

Method direct :

Examples :

$$(E_1)\ :\ \begin{cases}\frac{\partial^2 u}{\partial y^2}=ysin(x)\\ u(x,0)=x^2\\ u(x,2)=\frac{4}{3}\sin(x)\end{cases}$$

$$\frac{\partial^2 u}{\partial y^2} = ysin(x) \Rightarrow \frac{\partial u}{\partial y} = \int ysin(x)dy = \frac{1}{2}y^2 \sin(x) + f(x) \quad , \; f \text{ arbitrary Function .}$$

$$\Rightarrow u(x,y) = \int \left(\frac{1}{2}y^2 \sin(x) + f(x)\right) dy = \frac{1}{6}y^3 \sin(x) + yf(x) + g(x).$$

where $g$ is arbitrary function.

$$\begin{cases} u(x,0) = x^2 \\ u(x,2) = \frac{4}{3}\sin(x) \end{cases} \Rightarrow \begin{cases} g(x) = x^2 \\ f(x) = -\frac{1}{2}x^2 \end{cases} \Rightarrow u(x,y) = \frac{1}{6}y^3 \sin(x) + \left(1 - \frac{1}{2}y\right)x^2.$$

$$(E_2) : \begin{cases} \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} = -3 \\ u(0,y) = 0 \\ \frac{\partial u}{\partial x}(x,0) = x^2 \end{cases}$$

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} = -3 \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - u\right) = -3 \Rightarrow \frac{\partial u}{\partial x} - u = \int -3dy$$

$\frac{\partial u}{\partial x} - u = -3y + f(x)$ ODE linear , $\quad f$ arbitrary.

$$\begin{cases} u_0 = ke^x \\ u_p = 3y + e^x \int f(x)e^{-x}dx \end{cases} \Rightarrow u(x,y) = ke^x + 3y + e^x \int f(x)e^{-x}dx$$

$$\begin{cases} u(0,y) = 0 \\ \frac{\partial u}{\partial x}(x,0) = x^2 \end{cases} \Rightarrow \begin{cases} k + 3y + \int f(x)e^{-x}dx = 0 \; \dots\dots\dots\dots\dots (1) \\ ke^x + e^x \int f(x)e^{-x}dx + f(x) = x^2 \; \dots\dots (2) \end{cases}$$

$(2) - e^x(1) : \; f(x) = x^2 + ye^x$ et $\int f(x)e^{-x}dx = \int (x^2e^{-x} + y)dx = yx - (x^2 + 2x + 2)e^{-x}.$

$(1) \; : \; k = 2 - 3y.$ Then : $u(x,y) = (yx - 3y + 2)e^x + 3y - x^2 - 2x - 2 .$

Remark :

We can pose : $v = \dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

Method of charactéeristic :

To solve the PDE by this method we follow the following three steps :

First step : Find the solution of rhe following characteristic equation :

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0 \text{ where : } A, B \text{ and } C \text{ are cofficients of the PDE.}$$

We distinguish three cases : $\begin{cases} \Delta > 0 \ , \ \dfrac{dy}{dx} = \dfrac{B \pm \sqrt{\Delta}}{2A} \\ \Delta = 0 \ , \ \dfrac{dy}{dx} = \dfrac{B}{2A} \\ \Delta < 0 \ , \ \dfrac{dy}{dx} = \dfrac{B \pm i\sqrt{-\Delta}}{2A} \end{cases}$

Integrate the two differential equations, and put : $C_1 = \varphi(x,y)$ et $C_2 = \phi(x,y)$

Second step : Find the canonical form of the PDE.

Put the following change : $\begin{cases} s = C_1 = \varphi(x,y) \\ t = C_2 = \phi(x,y) \end{cases}$

Write PDE with the new variables $s$ and $t$.

Remark : The new equation is called a canonical form of PDE.

Third step : Solve the new PDE. And write the solution with the variables $x$ and $y$.

Examples :

$(E_1)$ : $\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial u}{\partial x} = 0$ Find canonical form of $(E_1)$.

Step 1: Solve the characteristic equation:

$$\left(\frac{dy}{dx}\right)^2 - 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}\left(\frac{dy}{dx} - 4\right) = 0 \Rightarrow \begin{cases} \dfrac{dy}{dx} = 0 \\ \dfrac{dy}{dx} = 4 \end{cases} \Rightarrow \begin{cases} y = C_1 \\ y = 4x + C_2 \end{cases} \Rightarrow \begin{cases} C_1 = y \\ C_2 = -4x + y \end{cases}$$

Step 2: The canonical form of $(E_1)$

Put : $\begin{cases} s = y \\ t = y - 4x \end{cases}$ , d'ou $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x} = -4\dfrac{\partial u}{\partial t} \\ \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = -4\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial t}\right) = 16\dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = -4\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial t}\right) = -4\dfrac{\partial^2 u}{\partial s \partial t} - 4\dfrac{\partial^2 u}{\partial t^2} \end{cases}$

$(E_1)$ : $\dfrac{\partial^2 u}{\partial s \partial t} + \dfrac{\partial u}{\partial t} = 0$ is the canonical form of $(E_1)$.

$(E_2)$ : $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$

Step 1 :

Characteristic equation : $\left(\dfrac{dy}{dx}\right)^2 - 1 = 0 \Rightarrow \dfrac{dy}{dx} = \pm 1.$

$$\begin{cases} \dfrac{dy}{dx} = 1 \\ \dfrac{dy}{dx} = -1 \end{cases} \Rightarrow \begin{cases} y = x + C_1 \\ y = -x + C_2 \end{cases} \Rightarrow \begin{cases} C_1 = y - x \\ C_2 = y + x \end{cases}$$

Step 2 : The canonical forme of $(E_2)$

Put : $\begin{cases} s = y - x \\ t = y + x \end{cases}$ , d'ou $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x} = -\dfrac{\partial u}{\partial s} + \dfrac{\partial u}{\partial t} \\ \dfrac{du}{dy} = \dfrac{du}{ds}\dfrac{ds}{dy} + \dfrac{du}{dt}\dfrac{dt}{dy} = \dfrac{du}{ds} + \dfrac{du}{dt} \end{cases}$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial u}{\partial s} + \dfrac{\partial u}{\partial t}\right) = \dfrac{\partial^2 u}{\partial s^2} - 2\dfrac{\partial^2 u}{\partial s \partial t} + \dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial s} + \dfrac{\partial u}{\partial t}\right) = \dfrac{\partial^2 u}{\partial s^2} + 2\dfrac{\partial^2 u}{\partial s \partial t} + \dfrac{\partial^2 u}{\partial t^2} \end{cases}$$

$\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = -4\dfrac{\partial^2 u}{\partial s \partial t} = 0 \Rightarrow \dfrac{\partial^2 u}{\partial s \partial t} = 0$ is the canonical form of $(E_2)$.

Step 3 :

$u = ?$ solution of $\dfrac{\partial^2 u}{\partial s \partial t} = 0$

$$\frac{\partial^2 u}{\partial s \partial t} = 0 \Rightarrow \frac{\partial u}{\partial s} = f(s) \Rightarrow u(x,y) = \int f(s)ds + gt) \ \Rightarrow u(x,y) = F(s) + g(t)$$

With: $F, g$ two arbitrary functions.

The general solution of $(E_2)$ is : $u(x,y) = F(y-x) + g(y+x)$.

$$(E_3):\ x^2 \frac{\partial^2 u}{\partial x^2} = y^2 \frac{\partial^2 u}{\partial y^2} \quad , \quad u = ?$$

Step 1 : Characteristic equation :

$$x^2\left(\frac{dy}{dx}\right)^2 - y^2 = 0 \ \Rightarrow \ \frac{dy}{dx} = \pm\left(\frac{y}{x}\right)^2.$$

$$\begin{cases} \frac{dy}{dx} = \frac{y}{x} \\ \frac{dy}{dx} = -\frac{y}{x} \end{cases} \Rightarrow \begin{cases} y = xC_1 \\ y = \frac{C_2}{x} \end{cases} \Rightarrow \begin{cases} C_1 = \frac{y}{x} \\ C_2 = xy \end{cases}$$

Step 2 : Canonical form of $(E_3)$

$$\text{Put : } \begin{cases} s = \frac{y}{x} \\ t = xy \end{cases} , \qquad \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = -\frac{y}{x^2}\frac{\partial u}{\partial s} + y\frac{\partial u}{\partial t} \\ \frac{du}{dy} = \frac{du}{ds}\frac{ds}{dy} + \frac{du}{dt}\frac{dt}{dy} = \frac{1}{x}\frac{\partial u}{\partial s} + x\frac{\partial u}{\partial t} \end{cases}$$

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial u}{\partial s} + y\frac{\partial u}{\partial t}\right) = \frac{y^2}{x^4}\frac{\partial^2 u}{\partial s^2} - 2\frac{y^2}{x^2}\frac{\partial^2 u}{\partial s \partial t} + y^2\frac{\partial^2 u}{\partial t^2} + 2\frac{y}{x^3}\frac{\partial u}{\partial s} \\ \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial u}{\partial s} + x\frac{\partial u}{\partial t}\right) = \frac{1}{x^2}\frac{\partial^2 u}{\partial s^2} + 2\frac{\partial^2 u}{\partial s \partial t} + x^2\frac{\partial^2 u}{\partial t^2} \end{cases}$$

$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow -4y^2\frac{\partial^2 u}{\partial s \partial t} + 2\frac{y}{x}\frac{\partial u}{\partial s} = 0 \Rightarrow -4st\frac{\partial^2 u}{\partial s \partial t} + 2s\frac{\partial u}{\partial s} = 0 .$$

$2\text{t}\,\frac{\partial^2 u}{\partial s \partial t} - \frac{\partial u}{\partial s} = 0$ is the canonical Form of $(E_3)$.

Step 3 : $u = ?$

$$\text{Put : } \frac{\partial u}{\partial s} = v \ \Rightarrow 2\,t\,\frac{\partial v}{\partial t} - v = 0 \Rightarrow \frac{\partial v}{v} = \frac{\partial t}{2t} \ \Rightarrow v = f(s)\sqrt{t}$$

$$\frac{\partial u}{\partial s} = v = f(s)\sqrt{t} \quad \Rightarrow u = \sqrt{t}\int f(s)ds + g(t) = \sqrt{t}\,F(s) + g(t)$$

Then : $u(x,y) = \sqrt{xy}\,F\left(\frac{y}{x}\right) + g(xy)$ Is the general solution of $(E_3)$.