TD 01

Exercise 1

Consider the following four statements :

(a) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0.$
(b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0.$
(c) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0.$
(d) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^2 > x.$

1. Are the statements a, b, c, d true or false ? Provide their negations.
2. Let P, Q, and R be three statements. Verify by creating a truth table :
   (a) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$,
   (b) $\overline{(P \Rightarrow Q)} \Leftrightarrow P \wedge \overline{Q}$.

Exercise 2

Let f function of $\mathbb{R}$ in $\mathbb{R}$. Translate the following expressions into quantifier terms :

1. f is bounded above.
2. f is bounded.
3. f is even.
4. f never equals zero.
5. f is periodic.
6. f is increasing.
7. f is not the zero function.
8. f attains all values in $\mathbb{N}$.

Exercise 3

Let $f : \mathbb{R} \to \mathbb{R}$. What is the difference in meaning between the two proposed statements ?

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y = f(x)$ and $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y = f(x)$.
2. $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y = f(x)$ and $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y = f(x)$.
3. $\forall x \in \mathbb{R}, \exists M \in \mathbb{R}, f(x) \leq M$ and $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) \leq M$.

Exercise 4

Show by recurrence that :

1. $\forall n \in \mathbb{N}^\star : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$
2. $\forall n \in \mathbb{N}$, $4^n + 6n - 1$ is a multiple of 9.

Exercise 5

By the absurd show that :
$\forall n \in \mathbb{N}$, n^2 even $\Rightarrow n$ is even.

Exercise 6

By contrapositive, show that

1. If $(n^2 - 1)$ is not divisible by 8 then n is even.
2. $\forall \varepsilon > 0, |x| \leq \varepsilon \Rightarrow x = 0$.

Solutions of TD 01

Solution of exercise 1

1. is false because its negation is $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $x+y \leq 0$, is true.
   let $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$, such that $x+y \leq 0$, for example we can take $y = -(x+1)$ and then $x+y = x-x-1 = -1 \leq 0$.
2. is true, for any given x for example we can take $y = -x+1$, and then $x+y = 1 \geq 0$. Then negation of **2)** is $\exists x \in \mathbb{R}\,;\, \forall y \in \mathbb{R}\,;\, x+y \leq 0$:
3. is false for example $x = -1$, $y = 0$. The negation is $\exists x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $x+y \leq 0$.
4. is true, we can take $x = -1$, the negation is $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $y^2 \leq x$.
5. Truth table :

P	Q	R	$P \vee Q$	$(P \vee Q) \wedge R$	$P \wedge R$	$Q \wedge R$	$(P \wedge R) \vee (Q \wedge R)$
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0
1	0	1	1	1	1	0	1
0	1	1	1	1	0	1	1
1	0	0	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	0	0	0	0	0

6.

P	Q	$\overline{Q}$	$P \wedge \overline{Q}$	$P \Rightarrow Q$	$\overline{P \Rightarrow Q}$
1	1	0	0	1	0
0	0	1	0	1	0
1	0	1	1	0	1
0	1	0	0	1	0

Solution of exercise 2

1. $\exists M \in \mathbb{R}$, $\forall x \in \mathbb{R}$, $f(x) \leq M$.
2. $\exists M \in \mathbb{R}$, $\exists m \in \mathbb{R}$, $\forall x \in \mathbb{R}$, $\quad m \leq f(x) \leq M$.
3. $\forall x \in \mathbb{R}$, $f(x) = f(-x)$.
4. $\forall x \in \mathbb{R}$, $f(x) \neq 0$.
5. $\exists \alpha \in \mathbb{R}^*$, $\forall x \in \mathbb{R}$, $f(x+\alpha) = f(x)$.
6. $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $x \leq y \Rightarrow f(x) \leq f(y)$.
7. $\exists x \in \mathbb{R}$, $f(x) \neq 0$.
8. $\forall n \in \mathbb{N}$, $\exists x \in \mathbb{R}$, $f(x) = n$.

Solution of exercise 3

1. The first statement is verified by any function, the second one means that f is constant.
2. The first statement means that f takes every value in $\mathbb{R}$, the second one is absurd.
3. The first statement is always verified, the second one means that f is bounded.

Solution of exercise 4

1. Let's show that $P_n : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}, \quad \forall n \in \mathbb{N}^\star$

   • For $n = 1$ we have : $1^3 = \frac{1^2(2)^2}{4} = 1$
   So P_1 is true.
   • We suppose that $P_n : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$ is true.
   And we show that : $1^3 + 2^3 + ... + n^3 + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}$ is true.
   Using P_n we obtain :

$$\begin{aligned} 1^3 + 2^3 + ... + (n+1)^3 &= 1^3 + 2^3 + ... + n^3 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} \end{aligned}$$

   Thus P_{n+1} is true, then $\forall n \in \mathbb{N}^\star : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$.

2. Let's show that $P_n : 4^n + 6n - 1, \quad \forall n \in \mathbb{N}$ is a multiple of 9,
   that's to say $\forall n \in IN$, $\exists k \in \mathbb{Z}$ / $4^n + 6n - 1 = 9k$
   • For $n = 1$ we have : $\exists k = 1 \in \mathbb{Z}$, $4 + 6 - 1 = 9 = 9(1)$, P_1 is true.
   • We suppose that : $\forall n \in IN$, $\exists k \in \mathbb{Z}$ / $4^n + 6n - 1 = 9k$ is true..
   And we show that : $\forall n \in IN$, $\exists ?k' \in \mathbb{Z}$ / $4^{n+1} + 6(n+1) - 1 = 9k'$is true..

$$\begin{aligned} 4^{n+1} + 6(n+1) - 1 &= 4.4^n + 6n + 6 - 1 \\ &= (9-5)4^n + 6n + 5 \\ &= 9(4^n) - 5.4^n - 5(6n) + 36n + 5 \\ &= -5(4^n + 6n - 1) + 9(4^n) + 36n \\ &= -5(9k) + 9(4^n) + 9(4n) \\ &= 9(-5k + 4^n + 4n) \\ &= 9k', \quad \text{such as } k' = -5k + 4^n + 4n \end{aligned}$$

Solution of exercise 5

Let $n \in \mathbb{N}$ by the absurd suppose that n^2 is even and n is odd, then $\exists k \in \mathbb{Z}$ such that :
$n = 2k + 1$ hence $n^2 = 2(2k^2 + 2k) + 1 = 2k' + 1$, $k' = (2k^2 + 2k) \in \mathbb{Z}$,
n^2 is odd contradiction because n^2 is even.
What we initially assumed is false i.e. $\forall n \in \mathbb{N}, n^2$ even $\Rightarrow n$ is even.

Solution of exercise 6

1. Let us show that its contrapositive : n is odd $\Rightarrow (n^2 - 1)$ is divisible by 8 is true
   Let n be odd then $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$ and therefore $n^2 = 4k^2 + 4k + 1$.
   $n^2 - 1 = 4k^2 + 4k = 4k(k+1)$ it is enough to show that $k(k+1)$ is even.
   Let us show that $k(k+1)$ is even, we have two cases :
   • If k is even then $k+1$ is odd so the product of an even number and an odd number is even.
   • If k is odd, then $k+1$ is even so the product is even, it's the same reasoning (you should know that the product

of two consecutive numbers is always even).
Thus $k(k+1)$ is even $\exists k^{'} \in \mathbb{Z} \ / \ k(k+1) = 2k^{'}$, hence $n^2 - 1 = 4(2k^{'}) = 8k^{'}$.
$n^2 - 1$ is divisible by 8.

2. Let us show that its contrapositive : $x \neq 0 \Rightarrow \exists \varepsilon > 0, \, |x| > \varepsilon$ is true
   Let $x \neq 0$, there exists $\varepsilon = \frac{x}{2} > 0$ such that $|x| > \frac{x}{2}$ because $x \neq 0$ hence the result.