Larbi Ben Mhidi University - Oum El Bouaghi-

Faculty of exact sciences, natural and life sciences

Department of Mathematics and Computer Science **Academic year: 2023/2024**

Level: 1st year MI **Duration:1h30**

Exam: Analysis 1

Exercise 1 (6 pts)

1) Let a and b be two numbers where $a \in \mathbb{Q}^+$, $b \in \mathbb{Q}^+$and $\sqrt{ab} \notin \mathbb{Q}^+$. Prove that $\sqrt{a}+\sqrt{b} \notin \mathbb{Q}^+$.

2) Let A be a subset of $\mathbb{R}$ bounded from above, we define the set $-A=\{-x; x \in A\}$.
Prove that: $\inf(-A) = -\sup A$.

3) Prove that: $\forall x \in \mathbb{R} : \operatorname{arc}\tan x + \operatorname{arc}\operatorname{cotan} x = \frac{\pi}{2}$.

4) Write the expression $L(x) = \sin^3 x \cos^3 x$ in linear form (Note that: $\sin x \cos x = \frac{1}{2}\sin 2x$).

Exercise 2 (7 pts)

Let f be a function defined in the interval $I = [2, +\infty[$ by $f(x) = \ln x - \frac{1}{x} + 2$.

1) a) Prove that the function f is strictly increasing on I.

b) Using the mean value theorem Prove that: $\forall a, b \in I: |f(b) - f(a)| \leq \frac{3}{4}|b-a|$.

2) Let $(v_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}}$ be a sequences defined by: $\forall n \in \mathbb{N}: \begin{cases} u_0 = 2 \\ u_{n+1} = f(u_n) \end{cases}$; $\begin{cases} v_0 = 3 \\ v_{n+1} = f(v_n) \end{cases}$

a) Study the monotonicity of the two sequences $(v_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}}$

b) Prove that: $\forall n \in \mathbb{N}: |v_{n+1} - u_{n+1}| \leq \frac{3}{4}|v_n - u_n|$.

c) Using the proof by induction, prove that: $\forall n \in \mathbb{N}: |v_n - u_n| \leq \left(\frac{3}{4}\right)^n$.

3) Prove that $(v_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are adjacent.

4) We put $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n = \ell$.

a) Prove that $\forall n \in \mathbb{N}: |\ell - u_n| \leq |v_n - u_n|$. b) Deduce a value rounded to 10^{-2} for ℓ.

Exercise 3 (7 pts) Let f be a function defined in $\mathbb{R}$ by $g(x) = \begin{cases} x^2 \sin\frac{1}{x}, x > 0 \\ e^{x^2} - \cos x, x \leq 0 \end{cases}$.

1) Examine the continuity of g over $\mathbb{R}$. 2) Using L'Hopital's rule, calculate $\lim\limits_{x\to 0} \frac{e^{x^2} - \cos x}{x}$.

3) Examine the derivability of g over $\mathbb{R}$. 4) Express $g'(x)$ in terms of x.

5) Is the function g of class C^1 on $\mathbb{R}$? justify your answer.

6) Using L'Hopital's rule, calculate $\lim\limits_{x\to+\infty} [g(x) - x]$, What do you conclude?

Good luck.

Ex1 (6 pts)

(1) Assume that $\sqrt{a}+\sqrt{b} \in \mathbb{Q}^+$, so $(\sqrt{a}+\sqrt{b})^2 \in \mathbb{Q}^+$

$\Rightarrow a+b+2\sqrt{ab} \in \mathbb{Q}^+ \Rightarrow \sqrt{ab} \in \mathbb{Q}^+ \Rightarrow$ contradiction with the assumption $\sqrt{ab} \notin \mathbb{Q}^+$

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therefore $\sqrt{a}+\sqrt{b} \notin \mathbb{Q}^+$

(2) Since A is bounded from above, then $\exists M \in \mathbb{R}$ s.t $M = \sup(A)$

that is, $\begin{cases} \forall x \in A : x \leq M \\ \forall \varepsilon > 0, \exists a \in A : M - \varepsilon < a \end{cases}$ 0,5

thus : $\begin{cases} \forall x \in A : -x \geq -M \\ \forall \varepsilon > 0, \exists a \in A : -M + \varepsilon > -a \end{cases}$ 0,5

hence $\operatorname{Inf}(-A) = -M = -\sup(A)$ 0,5

(3) Set $f(x) = \arctan(x) + \operatorname{arccotan}(x) \Rightarrow f'(x) = \frac{1}{x^2+1} + \frac{-1}{x^2+1} = 0$ 0,5

$\Rightarrow \forall x \in \mathbb{R} : f(x) = cte = f(0)$ 0,5

$\Rightarrow \forall x \in \mathbb{R} : \arctan(x) + \operatorname{arccotan}(x) = \arctan(0) + \operatorname{arccotan}(0)$

$= \frac{\pi}{2}$ 0,5

(4) Put $z = e^{2xi} = \cos(2x) + i\sin(2x) \Rightarrow \frac{1}{z} = \cos(2x) - i\sin(2x)$. ($\forall k \in \mathbb{Z}$)

0,5 $\Rightarrow \sin(2x) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, also we can easily get : $\sin(2kx) = \frac{1}{2i}\left(z^k - \frac{1}{z^k}\right)$

Now, $L(x) = \sin^3(x)\cos^3(x) = (\sin(x)\cos(x))^3 = \left(\frac{1}{2}\sin(2x)\right)^3 = \frac{1}{8}\sin^3(2x)$

0,5 $$= \frac{1}{8}\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^3 = \frac{-1}{64i}\left[\left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)\right]$$

0,5 $$= \frac{-1}{64i}\left[2i\sin(6x) - 6i\sin(2x)\right]$$

0,5 $$= \frac{-1}{32}\sin(6x) + \frac{3}{32}\sin(2x).$$

Exo 2 (7 pts.) $f(x) = \ln(x) - \frac{1}{x} + 2$.

① ⓐ $f'(x) = \frac{1}{x} + \frac{1}{x^2} > 0 \Rightarrow f(x) \nearrow$ (str.) 0,5

ⓑ Using the M.V.T for f in the interval $[a, b]$ / $a, b \in I$,

we get: $\exists c \in]a, b[$ s.t. $|f(b) - f(a)| \leq f'(c)|b - a|$ 0,5 ... (1)

since $c \in]a, b[\subset I = [2, +\infty[$, we find: $c > 2$, so $c^2 > 4$

thus, $\frac{1}{c} + \frac{1}{c^2} < \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ 0,5 $\Rightarrow f'(c) < \frac{3}{4} \Rightarrow |f(b) - f(a)| < \frac{3}{4}|b - a|$

for any $a, b \in I$. 0,5

② $\begin{cases} u_{n+1} = f(u_n) \\ u_0 = 2 \end{cases}$, $\begin{cases} v_{n+1} = f(v_n) \\ v_0 = 3 \end{cases}$

ⓐ since $f \nearrow$ (str.) and $u_0 = 2 < u_1 \approx 2{,}19 \Rightarrow (u_n) \nearrow$ (str.) 0,5

also $f \nearrow$ (str.) and $v_0 = 3 > v_1 \approx 2{,}77 \Rightarrow (v_n) \searrow$ (str.) 0,5

ⓑ Substituting $a = u_n$ and $b = v_n$ in the inequality (1), we get:

0,5 $|f(v_n) - f(u_n)| \leq \frac{3}{4}|v_n - u_n| \Rightarrow |v_{n+1} - u_{n+1}| \leq \frac{3}{4}|v_n - u_n|$; for any $n \in \mathbb{N}$

ⓒ For $n = 0$, we have: $|v_0 - u_0| = |3 - 2| = 1 \leq \left(\frac{3}{4}\right)^0$

Now, assume that: $|v_n - u_n| \leq \left(\frac{3}{4}\right)^n$ for $n \in \mathbb{N}$, we have: (1)

$|v_{n+1} - u_{n+1}| \leq \frac{3}{4}|v_n - u_n| \leq \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^n = \left(\frac{3}{4}\right)^{n+1}$ as requared.

③ Since $|v_n - u_n| \leq \left(\frac{3}{4}\right)^n$ and $\lim\left(\frac{3}{4}\right)^n = 0$, by squeez thm. we get:

0,5 $\lim(v_n - u_n) = 0$. Because $(u_n) \nearrow$, $(v_n) \searrow$ and $\lim(v_n - u_n) = 0$, we conclude that (u_n) and (v_n) are adjacent.

④ Since (u_n) and (v_n) are adjacent, we deduce that $\lim(v_n) = \lim(u_n) = \ell$ (say)

0,5

Thus $\forall n \in \mathbb{N}$: $u_n \leq \ell \leq v_n \Rightarrow \forall n \in \mathbb{N}: 0 \leq \ell - u_n \leq v_n - u_n \Rightarrow \forall n \in \mathbb{N}: |\ell - u_n| \leq |v_n - u_n|$ 0,5

⑤ By calculating the successive values of $|v_n - u_n|$, we get $|v_8 - u_8| < 0{,}01$

0,5

so, u_8 can be considered a rounded value to 10^{-2} for ℓ, thus $\ell \approx u_8 \approx 2{,}53$ 0,5

Ex3 (7pts) $g(x)=\begin{cases} x^2\sin\frac{1}{x}, & x>0 \\ e^{x^2}-\cos x, & x\le 0\end{cases}$

① g is continuous over $]-\infty,0[\cup]0,+\infty[$ (0,25)

$\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-}\left(e^{x^2}-\cos(x)\right) = g(0) = 0$ .. ① (0,5)

Since, for $x>0$; $0\le x^2\sin\frac{1}{x}\le x^2$, by squeez thm, we get:

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} x^2\sin\frac{1}{x} = 0 = g(0)$ -- ② (0,5)

from ① and ②, we find that $g(x)$ is continuous at $x=0$

② $\lim\limits_{x\to 0}\frac{e^{x^2}-\cos x}{x} = IF\frac{0}{0}$, using L'Hopital we get: $\lim\limits_{x\to 0}\frac{e^{x^2}-\cos x}{x} = \lim\limits_{x\to 0}\frac{(e^{x^2}-\cos x)'}{(x)'}$

$\Rightarrow \lim\limits_{x\to 0}\frac{e^{x^2}-\cos x}{x} = \lim\limits_{x\to 0}\frac{2xe^{x^2}+\sin x}{1} = 0$ (1)

(0,25)

③ g is diff$_{ble}$ over $]-\infty,0[\cup]0,+\infty[$. We have $g'_-(0) = \lim\limits_{x\to 0^-}\frac{g(x)-g(0)}{x-0}$

$\Rightarrow g'_-(0) = \lim\limits_{x\to 0^-}\left(\frac{e^{x^2}-\cos x}{x}\right) = 0$ (from qt, 2) --- ③ (0,5)

since, for $x>0$; $0\le x\sin\frac{1}{x}\le x$; by squeez thm, we get:

$g'_+(0) = \lim\limits_{x\to 0^+}\frac{g(x)-g(0)}{x-0} = \lim\limits_{x\to 0^+}\left(x\sin\frac{1}{x}\right) = 0$ -- ④ (0,5)

from ③ and ④, we get: $g'(0)=0$.

④ $g'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}; & x>0 \\ 0 & ; x=0 \\ 2xe^{x^2}+\sin x & ; x<0\end{cases}$ (1)

⑤ Since, for $x_n = \frac{1}{2\pi n+\frac{\pi}{2}}$ and $x'_n = \frac{1}{2\pi n+\frac{\pi}{3}}$, we have $\lim x_n = \lim x'_n = 0$

and $\lim g'(x_n) = \lim\left[\frac{1}{2\pi n+\frac{\pi}{2}}\sin\left(2\pi n+\frac{\pi}{2}\right) - \cos\left(2\pi n+\frac{\pi}{2}\right)\right] = 0$

(0,5) $\lim g'(x'_n) = \lim\left[\frac{1}{2\pi n+\frac{\pi}{3}}\sin\left(2\pi n+\frac{\pi}{3}\right) - \cos\left(2\pi n+\frac{\pi}{3}\right)\right] = -\frac{1}{2}$

$\neq \Rightarrow \nexists \lim\limits_{x\to 0} g'(x)$ (0,5) $\Rightarrow g \notin C^1(\mathbb{R})$

⑥ $\lim\limits_{x\to+\infty}(g(x)-x) = \lim\limits_{x\to+\infty}\left(x^2\sin\frac{1}{x} - x\right) = \lim\limits_{x\to+\infty}\frac{\left(\sin\frac{1}{x}-\frac{1}{x}\right)}{\left(\frac{1}{x^2}\right)} = IF\frac{0}{0}$ (0,5), using L'Hopital, we get:

$\lim\limits_{x\to+\infty}(g(x)-x) = \lim\limits_{x\to+\infty}\frac{\left(\sin\frac{1}{x}-\frac{1}{x}\right)'}{\left(\frac{1}{x^2}\right)'} = \lim\limits_{x\to+\infty}\frac{\left(-\frac{1}{x^2}\cos\frac{1}{x}+\frac{1}{x^2}\right)}{\left(-\frac{2}{x^3}\right)} = \lim\limits_{x\to+\infty}\frac{\left(-\cos\frac{1}{x}+1\right)}{\left(-\frac{2}{x}\right)} = IF\frac{0}{0}$, using L'Hop-

once more, we get: $\lim\limits_{x\to+\infty}(g(x)-x) = \lim\limits_{x\to+\infty}\frac{\left(\frac{1}{x^2}\sin\frac{1}{x}\right)}{\left(\frac{2}{x^2}\right)} = \lim\limits_{x\to+\infty}\frac{\sin\frac{1}{x}}{2} = 0$ (0,5)

$\Rightarrow$ so g accepts an asymptotic line in the neighborhood of $+\infty$ with eqn. ($y=x$) (0,5)