Chapter 03 : Differential equations

I- Ordinary differential equations

1. Definitions
2. First order differential equations
3. second order equations with constant coefficients

1. Ordinary differential equations

1) Definition: Every relation between real variable x, unknown continuous function y, and its Derivatives $y', y'', y^{(3)}, \dots, y^{(n)}$ is called an ordinary differential equation.

Any ordinary differential equation (ODE) is presented by one of the following:

$$(E) \;:\; F\left(x, y,\ y', \dots, y^{(n)}\right) = 0 \quad \text{ou} \quad y^{(n)} = F\left(x, y, y', \dots, y^{(n-1)}\right)$$

We call the integer n in equation (E) the order of the equation.

Examples :

$(E_1) : y' = xy + 3$ EDO of order 1.

$(E_2) : y'' + x^2 y' = x$ EDO of second order.

$(E_3) : x(y')^2 + y + e^x = 0$ EDO of first order.

$(E_4) : y'' - (y')^3 = \cos(x)$ EDO of order 2.

2) Solution of the ordinary differential equation:

We call solution of the equation (E) every function φ, n-times deffirentiable on $I \subseteq \mathbb{R}$, and $F\left(x, \varphi,\ \varphi', \dots, \varphi^{(n)}\right) = 0$.

We call φ the integral of the equation(E) on $I \subseteq \mathbb{R}$. And the graph of φ Is called the integral curve of (E).

Theorem:

Let the differential equation $(E) \;:\; F\left(x, y,\ y', \dots, y^{(n)}\right) = 0$

If φ_1, and φ_2 are two solutions of (E) on $I \subseteq \mathbb{R}$.

Then: For any $\alpha, \beta \in \mathbb{R}$, $\alpha\varphi_1 + \beta\varphi_2$ is solution of (E).

Example :

Let $(E) : y'' - 5y + 6y = 0$, $\varphi_1(x) = e^{2x}$, and $\varphi_2(x) = e^{3x}$are solutions of (E).

Then: $\varphi(x) = \varphi_1(x) + \varphi_2(x) = e^{2x} + e^{3x}$ is solution of(E).

$(\varphi'(x) = 2e^{2x} + 3e^{3x}$, $\varphi''(x) = 4e^{2x} + 9e^{3x}$, $\varphi''(x) - 5\varphi'(x) + 6\varphi(x) = 0$)

3) Cauchy problem :

The Cauchy problem is written by : (P) : $\begin{cases} y^{(n)} = F\left(x, y, y', \dots, y^{(n-1)}\right) \\ y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1} \end{cases}$

Example : Solve the problem (P) : $\begin{cases} y' + 2xy = 0 \\ y(0) = 2 \end{cases}$

We have $y' + 2xy = 0 \Rightarrow y' = -2xy \;\left(y' = \frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = -2xy$

$$\Rightarrow \frac{dy}{y} = -2xdx$$

$$\Rightarrow \int \frac{dy}{y} = -2\int xdx$$

$$\Rightarrow ln|y| = -2x + c$$

$$\Rightarrow y = e^{-2x+c} = e^c e^{-2x} = ke^{-2x} \quad (e^c = k)$$

We say $y = ke^{-2x}$ the general solution of (P) .

For $y(0) = 2$ we obtain $k = 2$. Then $y_p = 2e^{-2x}$ is called the particular solution of (P).

2. Ordinary differential equation of first order

They have the form : $f(x, y, y') = 0$ ou $y' = f(x, y)$

There are three main types of differential equations of first order.

Differential equation with separate variables.

Homogeneous differential equations.

Linear differential equations.

And a finite number of special equations: Bernoulli equation, equation of Riccati, equation Of Lagrange, and Clairaut equation.…

Resolution of differential equations of order 1 :

1) differential equations with separate variables: They have the form: $f(y)y' = g(x)$

$$\int f(y)dy = \int g(x)dx \quad \text{(Find } y \text{ as a function of } x\text{)}$$

Examples : 1) (E_1) : $(1 + x^2)y' = xy$ EDO with separate variables

$(1 + x^2)y' = xy \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{1+x^2} \Rightarrow \int \frac{dy}{y} = \int \frac{x}{1+x^2}dx \Rightarrow ln|y| = \frac{1}{2}ln(1 + x^2) + c$

$$\Rightarrow \ln|y| = ln\left|k\sqrt{1 + x^2}\right| \quad (c = \ln(k))$$

$$\Rightarrow y = k\sqrt{1 + x^2}$$

Then $y = k\sqrt{1 + x^2}$ is the general solution of (E_1).

2) $(E_2)\ :\ x\,y' = y + xy$

$$x\,y' = y + xy \;\Rightarrow xy' = y(1+x) \;\Rightarrow \frac{dy}{y} = \frac{1+x}{x}dx \;\Rightarrow ln|y| = x + ln|x| + c$$

$$\Rightarrow y = e^{x+ln|x|+c}$$

$$\Rightarrow y \;= kxe^{x} \qquad (k = e^{c}).$$

Then $\;y = kxe^{x}\;$ is the general solution of (E_2).

3) $(E_3)\ :\ \begin{cases} y' = 2x\sqrt{y-1} \\ \quad y(1) = 1 \end{cases}$

$$y' = 2x\sqrt{y-1} \;\Rightarrow \frac{y'}{2\sqrt{y-1}} = x \;\Rightarrow \sqrt{y-1} = \frac{1}{2}x^2 + c \;\Rightarrow y = 1 + \left(\frac{1}{2}x^2 + c\right)^2$$

Then $y = 1 + \left(\frac{1}{2}x^2 + c\right)^2$ is the general solution of(E_3) .

$y(1) = 1 \;\Rightarrow\; c = \frac{-1}{2}$ Then $y_p = 1 + \frac{1}{4}(x^2-1)^2$ is the particular solution of (E_3).

2) Homogeneous differential equations : They presented in the form : $y' = f\left(\frac{y}{x}\right)$

To solve this equation put the change of variable $z = \frac{y}{x}$;

$$z = \frac{y}{x} \;\Rightarrow y = xz \;\text{ et }\; y' = z + xz'.$$

$y' = f\left(\frac{y}{x}\right) \;\Rightarrow z + xz' = f(z) \;\Rightarrow \frac{1}{f(z)-z}z' = \frac{1}{x}$ Equation with separate. variables.

Examples :

1) $(E_1)\ :\ (x^2 + y^2) - xyy' = 0$. Divide by x^2 .

The equation (E_1) becomes $\left(1 + \left(\frac{y}{x}\right)^2\right) - \frac{y}{x}y' = 0 \Rightarrow y' = \frac{1+\left(\frac{y}{x}\right)^2}{\frac{y}{x}} = f\left(\frac{y}{x}\right)$ (*)

Put : $z = \frac{y}{x} \;\Rightarrow y = xz \;\text{ et }\; y' = z + xz'$replace in the equation (*)

$$\frac{1}{f(z)-z}z' = \frac{1}{x} \;\Rightarrow\; zz' = \frac{1}{x} \;\Rightarrow\; \frac{1}{2}z^2 = \ln(x) + c \;\Rightarrow\; z^2 = ln(x^2) + k \quad (k = 2c)$$

$$\Rightarrow z = \pm\sqrt{ln(x^2) + k}$$

Or $\quad z = \pm\sqrt{ln(k_1x^2)} \quad (k = ln(k_1))$

$y = xz \;\Rightarrow y = \pm\, x\,\sqrt{ln(k_1x^2)}$ is the general solution of (E_1).

2) $(E_2)\ :\ xy' = y + x\cos\left(\frac{y}{x}\right)$ Divide by x . $y' = \frac{y}{x} + \cos\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right)$.

Put : $z = \frac{y}{x} \Rightarrow y = xz\,,\ y' = z + xz'$ et $\frac{z'}{f(z)-z} = \frac{1}{x} \Rightarrow \frac{1}{\cos(z)}z' = \frac{1}{x}$

$\int \frac{dz}{\cos(z)} = \int \frac{dx}{x}$ Put : $t = \tan\left(\frac{z}{2}\right), z = 2\arctan(t)\,, dz = \frac{2}{1+t^2}dt$

$\cos(z) = \frac{1-t^2}{1+t^2}$. $\int \frac{dz}{\cos(z)} = \int \frac{2}{1-t^2}dt = \int \left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt = \ln\left|\frac{1+t}{1-t}\right|$

$\int \frac{dx}{x} = \ln|x| + c = \ln|kx| \quad (c = \ln|k|)$

$\ln\left|\frac{1+t}{1-t}\right| = \ln|kx| \ \Rightarrow \frac{1+t}{1-t} = kx \ \Rightarrow t = \frac{kx-1}{kx+1} \ \Rightarrow z = 2\arctan\left(\frac{kx-1}{kx+1}\right)$

Et $y = 2x\arctan\left(\frac{kx-1}{kx+1}\right)$ is the general solution of (E_2).

3) $(E_3)\ :\ \begin{cases} x^2y' - 2xy + y^2 = 0 \\ y(1) = 2 \end{cases}$

$x^2y' - 2xy + y^2 = 0 \ \Rightarrow y' = 2\frac{y}{x} - \left(\frac{y}{x}\right)^2 = f\left(\frac{y}{x}\right)$ Put: $z = \frac{y}{x}\,, y = xz$

$y' = z + xz' = 2z - z^2 \Rightarrow \frac{1}{z-z^2}z' = \frac{1}{x} \Rightarrow \int \frac{dz}{z-z^2} = \int \frac{dx}{x}$

$\int \frac{dz}{z-z^2} = \int \left(\frac{1}{z} + \frac{1}{1-z}\right)dz = \ln\left|\frac{z}{1-z}\right| = \ln|kx| \ \Rightarrow z = \frac{kx}{kx+1} \ \Rightarrow y = \frac{kx^2}{kx+1}$

$y = \frac{kx^2}{kx+1}$ General solution of (E_3).

$y(1) = 2 \ \Rightarrow \frac{k}{k+1} = 2 \Rightarrow k = -2 \Rightarrow y_p = \frac{2x^2}{2x-1}$ particular solution of (E_3).

3) Linear differential equations: $(E)\ :\ y' + a(x)y = b(x)$,
a, b Two continuous functions on $I \subseteq \mathbb{R}$.

The lenear equation solved by two steps

1st Step : Find y_0 Solution of the equation (E_0) without the second member $(b(x) = 0)$.

$(E_0)\ :\ y' + a(x)y = 0 \ \Rightarrow \frac{y'}{y} = -a(x) \ \Rightarrow y_0 = k\,e^{-\int a(x)dx}$

2nd Step : Find y_p the particular solution of (E).

Use the method of constant variation. $k = k(x)$ (Function)

We have : $y = ke^{-\int a(x)dx} \Rightarrow y' = k' e^{-\int a(x)dx} - k\, a(x)e^{-\int a(x)dx}$

Replace in (E) : $k'e^{-\int a(x)dx} = b(x) \Rightarrow k = \int b(x)e^{\int a(x)dx}dx.$

Then : $y_p = ke^{-\int a(x)dx} = e^{-\int a(x)dx} \int b(x)e^{\int a(x)dx}dx.$

The general solution of (E) is given by : $y_G = y_0 + y_p.$

$y_G = k\, e^{-\int a(x)dx} + e^{-\int a(x)dx} \int b(x)e^{\int a(x)dx}dx.$

Examples :

1) $(E_1) : xy' + 2y = \frac{1}{2}x^3$

$xy' + 2y = \frac{1}{2}x^3 \Rightarrow y' + \frac{2}{x}y = \frac{1}{2}x^2 \qquad a(x) = \frac{2}{x}\ , b(x) = \frac{1}{2}x^2$

$i)\ y_0 =? \quad (E_0) : y' + \frac{2}{x}y = 0 \qquad y_0 = k\, e^{-\int a(x)dx}$

$\int a(x)dx = \int \frac{2}{x}dx = 2\ln(x) = ln(x^2)$ Alors : $y_0 = \frac{k}{x^2}.$

$ii)\ y_p =?$ Particular solution of (E_1)

$y_p = e^{-\int a(x)dx} \int b(x)e^{\int a(x)dx}dx = \frac{1}{x^2}\int \frac{1}{2}x^4 dx = \frac{1}{10}x^3.$

Conclusion : $y_G = y_0 + y_p = \frac{k}{x^2} + \frac{1}{10}x^3$

2) $(E_2) : \begin{cases} y' + 2y = e^x \\ y(0) = 1 \end{cases}$ $a(x) = 2,\ b(x) = e^x$

$i)\ y_0 = ?\quad (E_0) :\ y' + 2y = 0 \qquad y_0 = k\, e^{-\int a(x)dx}$

$\int a(x)dx = 2\int dx = 2x$ Alors : $y_0 = ke^{-2x}$.

$ii)\ y_p = ?$ Particular solution of (E_2)

$y_p = e^{-\int a(x)dx}\int b(x)e^{\int a(x)dx}dx = e^{-2x}\int e^{3x}dx = \frac{1}{3}e^x.$

Conclusion : $y_G = y_0 + y_p = ke^{-2x} + \frac{1}{3}e^x.$

$y(0) = 1 \Rightarrow k + \frac{1}{3} = 1 \Rightarrow k = \frac{2}{3}.\ \ y = \frac{2}{3}e^{-2x} + \frac{1}{3}e^x.$

3) $(E_3) : y' - \frac{y}{x} = x\arctan(x) \qquad a(x) = -\frac{1}{x}\ ,\ b(x) = x\arctan(x)$

$i)\ y_0 = ?\quad (E_0) :\ y' - \frac{1}{x}y = 0 \qquad y_0 = k\, e^{-\int a(x)dx}$

$\int a(x)dx = -\int \frac{dx}{x} = -ln|x| \Rightarrow y_0 = k\,x.$

$ii)\ y_p = ?$ Particular solution of (E_3)

$y_p = e^{-\int a(x)dx}\int b(x)e^{\int a(x)dx}dx = x\int \arctan(x)\,dx$ Par partie

$\int \arctan(x)\,dx = xarctan(x) - \int \frac{x}{1+x^2}dx = xarctan(x) - \frac{1}{2}ln(1+x^2)$

$y_p = x^2arctan(x) - \frac{1}{2}\,x\,ln(1+x^2)$

Conclusion : $y_G = k\,x + x^2arctan(x) - \frac{1}{2}\,x\,ln(1+x^2).$

4) Differential equation of Bernoulli :

They take the form: $(E)\ :\ y' + a(x)y = b(x)y^n,\ n > 1$ (integer)

To solve the Bernoulli equation put the change of variable: $z = \frac{1}{y^{n-1}}$.

$z = \frac{1}{y^{n-1}} \Rightarrow z' = \frac{1-n}{y^n}y'$ Replace in (E)

$(E): \frac{1}{y^n}y' + \frac{a(x)}{y^{n-1}} = b(x) \Rightarrow \frac{1}{1-n}z' + a(x)z = b(x)$ is linear ODE.

Examples :

1) $(E_1) : xy' + y = y^2\ln(x)$ EDO of Bernoulli with $n = 2$.

$$xy' + y = y^2\ln(x) \Rightarrow \frac{y'}{y^2} + \frac{1}{xy} = \frac{ln(x)}{x}$$

Put : $z = \frac{1}{y}$, $z' = \frac{-1}{y^2}y' \implies (E_1) : -z' + \frac{1}{x}z = \frac{\ln(x)}{x}$ Linear EDO with :

$a(x) = \frac{-1}{x}$, $b(x) = \frac{-\ln(x)}{x}$

$i)$ $z_0 =?$ solution of $z' - \frac{1}{x}z = 0,\quad z_0 = k\, e^{-\int a(x)dx} = k\, x$

$ii)$ $z_p =?$ Particular solution of (E_1).

$z_p = e^{-\int a(x)dx}\int b(x)e^{\int a(x)dx}dx = -x\int\frac{\ln(x)}{x^2}dx$ By parts.

$$\int\frac{\ln(x)}{x^2}dx = \frac{-1}{x}(1+\ln(x)) \Rightarrow z_p = 1 + \ln(x)$$

Then : $z_G = z_0 + z_p = k\,x + \ln(x) + 1$

$z = \frac{1}{y} \Rightarrow y = \frac{1}{z} = \frac{1}{kx+\ln(x)+1}$ Is the general solutionof (E_1).

2) $(E_2) : x^2y' + xy = y^5$ EDO of Bernoulli with $n = 5$.

$x^2y' + xy = y^5 \Rightarrow \frac{y\prime}{y^5} + \frac{1}{x}\frac{1}{y^4} = \frac{1}{x^2}$ Put : $z = \frac{1}{y^4}$, $z' = \frac{-4}{y^5}y'$ the equation becomes

$\frac{-1}{4}z' + \frac{1}{x}z = \frac{1}{x^2} \Rightarrow z' - \frac{4}{x}z = \frac{-4}{x^2}$ ODE linear

$i)$ z_0? Solution of the equation $z' - \frac{4}{x}z = 0$

$z' - \frac{4}{x}z = 0 \Rightarrow z_0 = kx^4$.

$ii)$ $z_p =?$ Particular solution of $z' - \frac{4}{x}z = \frac{-4}{x^2}$. By variation of the constant $k = k(x)$ (Function)

We have $z = kx^4$ et $z' = k'x^4 + 4kx^3$ Replace in $z' - \frac{4}{x}z = \frac{-4}{x^2}$ we find :

$k' = \frac{-4}{x^6} \Rightarrow k = \frac{4}{5x^5}$ Alors : $z_p = \frac{4}{5\,x}$ et $z_G = z_0 + z_p = kx^4 + \frac{4}{5\,x}$.

$$z = \frac{1}{y^4} \Rightarrow y = \pm \frac{1}{\sqrt[4]{kx^4 + \frac{4}{5\,x}}}.$$

5) Differential equation of Riccati :

They have the form : $(E) : y' + a(x)y^2 + b(x)y = c(x)$ with : $a, b,$ and c three continuous functions on the interval $I \subseteq \mathbb{R}$.

To solve the Riccati equation, we need to know a particular solution y_1 , and put $y = y_1 + \frac{1}{z}$.

$y = y_1 + \frac{1}{z}$, et $y' = y'_1 - \frac{1}{z^2} z'$ Replace in(E), we get the following linear equation :

$$z' - (2a(x)y_1 + b(x))z = a(x).$$

Examples :

1) $(E) : \begin{cases} x^2y' = xy - x^2y^2 - 1 \\ y(1) = 2 \end{cases}$ with $y_1 = \frac{1}{x}$ is a particular solution.

$x^2y' = xy - x^2y^2 - 1 \Rightarrow y' + y^2 - \frac{1}{x}y = \frac{-1}{x^2}$ Equation of Riccati

with : $a(x) = 1,\ b(x) = \frac{-1}{x}$, et $c(x) = \frac{-1}{x^2}$.

To solve it put : $y = \frac{1}{z} + y_1$, $y' = \frac{-1}{z^2} z' + y'_1$ We get : $z' - \frac{1}{x} z = 1$ (linear)

$i)$ $z_0 = ?$ Solution of $z' - \frac{1}{x} z = 0$, $z_0 = kx$

$ii)$ $z_p = ?$ a particular solution of $z' - \frac{1}{x} z = 1.$ $z_p = x\ln|x|$

Then: $z_G = z_0 + z_p = kx + x\ln|x|$ a general solution of linear equation.

And $y = \frac{1}{z} + y_1 = \frac{1}{kx + x\ln|x|} + \frac{1}{x}$ Is the solution of the equation (E).

We have $y(1) = 1 \Rightarrow \frac{1}{k} + 1 = 2 \Rightarrow k = 1$ Then : $y_2 = \frac{1}{x + x\ln|x|} + \frac{1}{x} = \frac{2 + \ln|x|}{x + x\ln|x|}$

2) $(E) : y' = -2x + \left(1 + \frac{1}{x}\right) y + \frac{1}{x} y^2$, $y_0 = x$ The particular solution.

$$y' = -2x + \left(1 + \frac{1}{x}\right) y + \frac{1}{x} y^2 \Rightarrow y' - \frac{1}{x} y^2 - \left(1 + \frac{1}{x}\right) y = -2x$$

$$a(x) = -\frac{1}{x}\ ,\quad b(x) = -\left(1 + \frac{1}{x}\right)\ ,\quad c(x) = -2x$$

Put : $y = \frac{1}{z} + x$, $y' = \frac{-1}{z^2} z' + 1$ (E) Becomes : $z' + \left(3 + \frac{1}{x}\right) z = \frac{-1}{x}$ (linear equation)

$i)$ $z_0 = ?$ Solution of $z' + \left(3 + \frac{1}{x}\right) z = 0,$ $z_0 = \frac{k}{x} e^{-3x}$

$ii)$ z_p =? Particular solution of $z' + \left(3+\frac{1}{x}\right)z = \frac{-1}{x}$, $z_p = \frac{-1}{3\,x}$

$z_G = z_0 + z_p = \frac{k}{x}e^{-3x} - \frac{1}{3\,x}$ General solution of the equation $z' + \left(3+\frac{1}{x}\right)z = \frac{-1}{x}$

And $y = \frac{1}{z} + x = \frac{3x}{3ke^{-3x}-1} + x$ (you can replace $3k$ by k).

$$= \frac{3x}{ke^{-3x}-1} + x$$

3) Différential equations of second order :

The différential equations of order 2 have the form : $y'' = F(x,y,y')$ or $F(x,y,y',y'') = 0$.

There is two principal types of differential equations of order 2 :

Incomplete differential equations And Linear differential equations

1) Incomplete differential equations : Generally, There are three cases :

First case : $(E)\ :\ y'' = f(x)$

To solve it integrate two times.

$y'' = f(x) \Rightarrow y' = \int f(x)dx = F(x) + c$, F primitive de f.

$\Rightarrow y = \int (F(x)+c)dx = G(x) + cx + k$, G primitive of F.

Example : $(E)\ :\ y'' = \frac{1}{1+x}$

$y'' = \frac{1}{1+x} \Rightarrow y' = ln|1+x| + c \Rightarrow y = (1+x)ln|1+x| - x + cx + k$

$y = (1+x)ln|1+x| + Cx + k \qquad (C = c-1)$

second case : $(E)\ :\ y'' = f(x,y')$

To solve it put the change : $y' = z$

$y' = z \Rightarrow z' = f(x,z)$ EDO of order one.

Example : $(E)\ :\ xy'' - y' = 0$ Put : $y' = z$ $(y'' = z')$

$xy'' - y' = 0 \Rightarrow xz' - z = 0 \Rightarrow \frac{z'}{z} = \frac{1}{x} \Rightarrow z = kx \Rightarrow y = \frac{1}{2}kx^2.$

Third case : $(E)\ :\ y'' = f(y,y')$

Put: $u(y) = y'$, u function on y

$u(y) = y' \Rightarrow y'' = u'(y)y' = uu'$

(E) becomes : $uu' = f(y,u)$ EDO of order 1 (the variable is y)

Example : (E) : $y'' = \frac{y'^2}{\tan(y)}$

Put : $u = u(y) = y' \Rightarrow uu' = \frac{u^2}{\tan(y)} \Rightarrow \frac{1}{u}u' = \frac{\cos(y)}{\sin(y)}$

$\frac{1}{u}u' = \frac{\cos(y)}{\sin(y)} \Rightarrow ln|u| = ln|k\sin(y)| \Rightarrow u = ksin(y)$

$u = y' = ksin(y) \Rightarrow \int \frac{dy}{sin(y)} = \int kdx \Rightarrow kx + C = \int \frac{dy}{sin(y)}$ (Posons : $t = tan\left(\frac{y}{2}\right)$)

$\int \frac{dy}{sin(y)} = ln\left|tan\left(\frac{y}{2}\right)\right| = kx + C \Rightarrow tan\left(\frac{y}{2}\right) = C_1e^{kx}$. $(C_1 = e^C)$

$y = 2\,arctan(C_1e^{kx})$ Solution of (E).

2) Linear differential equations : $(E) : a(x)y'' + b(x)y' + c(x)y = f(x)$
With a, b, c, and f continuous functions on the interval $I \subseteq \mathbb{R}$.

Remark : In this chapter we interested a equations with a constant coefficients (a, b, et $\in \mathbb{R}$)

3) Linear differential equations of second order with a constant coefficients :

$$(E) : ay'' + by' + cy = f(x)$$

With a, b, c real numbers, $a \neq 0$, and f a continuous function on $I \subseteq \mathbb{R}$.

i) Differential equations of second order homogeneous: (without a second member)

$(E_0) : ay'' + by' + cy = 0$, a, b, c des réels , et $a \neq 0$.

To solve it consider the solution $y = e^{rx}$ with $r \in \mathbb{C}$.

$y = e^{rx}$, $y' = re^{rx}$, et $y'' = r^2e^{rx}$ Replace in the equation(E_0), we find

$ar^2 + br + c = 0$.

We call the polynomial $P(r) = ar^2 + br + c$, a characteristic polynomial associated with equation (E_0).

Find the general solution of the homogeneous equation (E_0) we distinguish three cases :

First case : $\Delta = b^2 - 4ac > 0$.

P have two real roots: $r_1 = \frac{-b+\sqrt{\Delta}}{2\,a}$, $r_2 = \frac{-b-\sqrt{\Delta}}{2\,a}$

And the general solution is given by : $y_0 = C_1e^{r_1x} + C_2e^{r_2x}$

Examples :

1) (E_0) : $y'' + y' - 2y = 0$ $P(r) = ar^2 + br + c = r^2 + r - 2$
$\Delta = 9 > 0$, $r_1 = -2$, $r_2 = 1$

Then : $y_0 = C_1e^{-2x} + C_2e^x$ is general solution of (E_0).

2) (E_0) : $3y'' - 5y' + 2y = 0$ $P(r) = 3r^2 - 5r + 2$

$\Delta = 1 > 0$, $r_1 = \frac{2}{3}$, $r_2 = 1$

Then : $y_0 = C_1 e^{\frac{2}{3}x} + C_2 e^x$

Second case : $\Delta = b^2 - 4ac = 0$.

P have one real root : $r_1 = r_2 = r = \frac{-b}{2a}$

The general solution given by: $y_0 = (C_1 x + C_2)e^{rx}$.

Examples :

1) (E_0) : $y'' + 2y' + y = 0$ $\quad$ $P(r) = r^2 + 2r + 1 = (r+1)^2$ $\quad$ $r = -1$

Then : $y_0 = (C_1 x + C_2)e^{-x}$

2) (E_0) : $9y'' - 6y' + y = 0$ $\quad$ $P(r) = 9r^2 - 6r + 1$ $\quad$ $\Delta = 0$, $r = \frac{1}{3}$

Then : $y_0 = (C_1 x + C_2)e^{\frac{1}{3}x}$

Third case : $\Delta = b^2 - 4ac < 0$.

P have two complex roots : $r_1 = \alpha + i\beta$, $r_2 = \overline{r_1} = \alpha - i\beta$

The general solution given by : $y_0 = \left(C_1 cos(\beta x) + C_2 sin(\beta x)\right)e^{\alpha x}$

Examples :

1) (E_0) : $y'' - 2y' + 2y = 0$ $\quad$ $P(r) = r^2 - 2r + 2$ $\quad$ $\delta = -1 < 0$ $(\Delta = -4)$

$r_1 = 1 + i$, $r_2 = \overline{r_1} = 1 - i$ $(\alpha = 1,\ \beta = 1)$

Then : $y_0 = \left(C_1 cos(x) + C_2 sin(x)\right)e^x$

2) (E_0) : $y'' + 2y = 0$ $\quad$ $P(r) = r^2 + 2 = (r - i\sqrt{2})(r + i\sqrt{2})$

$r_1 = i\sqrt{2}$, $r_2 = -i\sqrt{2}$ $\quad$ $(\alpha = 0$, $\beta = \sqrt{2})$

Then : $y_0 = C_1 cos(\sqrt{2}\,x) + C_2 sin(\sqrt{2}\,x)$

ii) Differential equation of second order with a second member:

(E) : $ay'' + by' + cy = f(x)$ with a, b, c a real , and $a \neq 0$.

The equation (E) solved in two steps :

First step: Find y_0 Solution of homogeneous equation (E_0) : $ay'' + by' + cy = 0$.

Second step: Find y_p particular solution of (E).

$y_p = C_1 y_1 + C_2 y_2$ with y_1 and y_2 are solutions of the homogeneous equation (E_0).

Determine C_1 et C_2 By the constant variation method.

Integrate C'_1 and C'_2 Solution of system : $\begin{cases} C'_1 y_1 + C'_2 y_2 = 0 \\ C'_1 y'_1 + C'_2 y'_2 = \frac{1}{a} f(x) \end{cases}$

Conclusion : $y_G = y_0 + y_p$ general solution of (E).

Examples :

1) $(E): 2y'' - 3y' + y = e^x$

$i)\ y_0 = ?$ Solution of $(E_0):\ 2y'' - 3y' + y = 0$

$P(r) = 2r^2 - 3r + 1$, $\Delta = 1 > 0$, $r_1 = 1$, $r_2 = \frac{1}{2}$

then : $y_0 = C_1 e^x + C_2 e^{\frac{1}{2}x}$

$ii)\ y_p = ?$ particular solution of (E). constant variation method. $C_1 = C_1(x)$ et $C_2 = C_2(x)$.

$$y_p = C_1 y_1 + C_2 y_2 = C_1 e^x + C_2 e^{\frac{1}{2}x}$$

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y'_1 + C_2' y'_2 = \frac{1}{a} f(x) \end{cases} \Rightarrow \begin{cases} C_1' e^x + C_2' e^{\frac{1}{2}x} = 0 & (1) \\ C_1' e^x + \frac{1}{2} C_2' e^{\frac{1}{2}x} = \frac{1}{2} e^x & (2) \end{cases}$$

$(2) - (1)$: $C_2' = -e^{\frac{1}{2}x} \Rightarrow C_2 = -\int e^{\frac{1}{2}x} dx = -2e^{\frac{1}{2}x}$.

$(1):\ C_1' = 1 \Rightarrow C_1 = x$.

Then : $y_p = (x-2)e^x$

Conclusion : $y_G = y_0 + y_p = (x - 2 + C_1)e^x + C_2 e^{\frac{1}{2}x}$. (We can take $C_3 = -2 + C_1$).

$$y_G = y_0 + y_p = (x + C_3)e^x + C_2 e^{\frac{1}{2}x}$$

2) $(E):\ y'' + y = \sin(x)$

$i)\ y_0 = ?$ Solution of $(E_0):\ y'' + y = 0$

$P(r) = r^2 + 1 = (r - i)(r + i)$ $r_1 = i$, $r_2 = -i$ $\alpha = 0$ et $\beta = 1$

Then : $y_0 = C_1 \cos(x) + C_2 \sin(x)$

$ii)\ y_p = ?$ particular solution of (E). constant variation method. $C_1 = C_1(x)$ et $C_2 = C_2(x)$.

$$y_p = C_1 y_1 + C_2 y_2 = C_1 \cos(x) + C_2 \sin(x)$$

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y'_1 + C_2' y'_2 = \frac{1}{a} f(x) \end{cases} \Rightarrow \begin{cases} C_1' cos(x) + C_2' \sin(x) = 0 & (1) \\ -C_1' \sin(x) + C_2' \cos(x) = \sin(x) & (2) \end{cases}$$

$(1) \times \sin(x) + (2) \times \cos(x)$: $C_2' = \sin(x)\cos(x) = \frac{1}{2}\sin(2x)$

$$C_2 = \frac{-1}{4}\cos(2x) = \frac{1}{2}\sin^2(x)$$

$$(1):\ C_1' = -\sin^2(x) = \frac{1}{2}(\cos(2x)-1) \Rightarrow C_1 = \frac{1}{4}\sin(2x) - \frac{1}{2}x$$

$$\text{Then : } y_p = \frac{1}{2}\sin(x)\cos^2(x) - \frac{1}{2}x\cos(x) + \frac{1}{2}\sin^3(x)$$

$$= \frac{1}{2}\sin(x) - \frac{1}{2}x\cos(x).$$

Conclusion : $y_G = y_0 + y_p = \left(C_1 - \frac{1}{2}x\right)\cos(x) + C_3\sin(x) \quad \left(C_3 = \frac{1}{2} + C_2\right).$

3) $(E):\begin{cases} 4y'' - 4y' + y = xe^{\frac{1}{2}x} \\ y(0) = 2\ ,\ y'(0) = 1 \end{cases}$

$i)$ $y_0 = ?$ Solution of (E_0) : $4y'' - 4y' + y = 0$

$$P(r) = 4r^2 - 4r + 1 \qquad \Delta = 0 \qquad r_1 = r_2 = r = \frac{1}{2}$$

$$\text{Then : } y_0 = (C_1x + C_2)e^{\frac{1}{2}x}$$

$ii)$ $y_p = ?$ particular solution of (E). constant variation method. $C_1 = C_1(x)$ et $C_2 = C_2(x)$.

$$y_p = (C_1x + C_2)e^{\frac{1}{2}x} = C_1xe^{\frac{1}{2}x} + C_2e^{\frac{1}{2}x}$$

We have $\begin{cases} C_1'xe^{\frac{1}{2}x} + C_2'e^{\frac{1}{2}x} = 0 \\ \left(\frac{1}{2}x+1\right)C_1'e^{\frac{1}{2}x} + \frac{1}{2}C_2'e^{\frac{1}{2}x} = \frac{1}{4}xe^{\frac{1}{2}x} \end{cases} \Rightarrow \begin{cases} C_1'x + C_2' = 0 & (1) \\ \left(\frac{1}{2}x+1\right)C_1' + \frac{1}{2}C_2' = \frac{1}{4}x & (2) \end{cases}$

$$(2)\times 2 - (1):\ C_1' = \frac{1}{4}x \Rightarrow C_1 = \frac{1}{8}x^2$$

$$(1):\ C_2' = -C_1'x \Rightarrow C_2' = \frac{-1}{4}x^2 \Rightarrow C_2 = \frac{-1}{12}x^3.$$

$$\text{Then : } y_p = \frac{1}{24}x^3e^{\frac{1}{2}x}$$

Conclusion : $y_G = \left(\frac{1}{24}x^3 + C_1x + C_2\right)e^{\frac{1}{2}x}.$

We have $\begin{cases} y(0) = 2 \\ y'(0) = 1 \end{cases} \Rightarrow \begin{cases} C_2 = 2 \\ C_1 + \frac{1}{2}C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_2 = 2 \\ C_1 = 0 \end{cases}$ Then : $y = \left(\frac{1}{24}x^3 + 2\right)e^{\frac{1}{2}x}.$

Remark : (Find y_p by substitution)

1) If the second member $f(x) = Q(x)e^{sx}$ with: Q polynômial and $s \in \mathbb{R}$.

we distinguish three cases :

$i)$ If s is not root of $P(r)$. Then : $y_p = R(x)e^{sx}$, R polynômial and $d°R = d°Q$

$ii)$ If s is simple root of $P(r)$. Then : $y_p = R(x)e^{sx}$ with $d°R = d°Q + 1$

Or $y_p = x\,R(x)e^{sx}$ with $d°R = d°Q$

$iii)$ If s is double root of $P(r)$. Then: $y_p = R(x)e^{sx}$ with $d°R = d°Q + 2$

Or $y_p = x^2\,R(x)e^{sx}$ avec $d°R = d°Q$

2) If the second member $f(x) = A\,cos(\beta x) + B\,sin(\beta x)$
we distinguish two cases :

$i)$ If βi is not root of $P(r)$. Then : $y_p = A_1\cos(\beta x) + B_1\sin(\beta x)$

$ii)$ If βi is a root of $P(r)$. Then : $y_p = x(A_1\cos(\beta x) + B_1\sin(\beta x))$.

Examples :

1) $(E) : y'' - 3y' + 2y = (2x^2 - 3)e^{-x}$

$i)$ $y_0 = ?$ Solution of (E_0) : $y'' - 3y' + 2y = 0$ $P(r) = r^2 - 3r + 2 = (r-1)(r-2)$

Then : $y_0 = C_1e^x + C_2e^{2x}$

$ii)$ $y_p = ?$ we have $f(x) = Q(x)e^{s\,x} = (2x^2 - 3)e^{-x}$, $s = -1$ and $d°Q = 2$

$s = -1$ is not root of P . Then: $y_p = R(x)e^{-x}$ $d°R = d°Q = 2$

$y_p = (ax^2 + bx + c)e^{-x}$

$y_p' = (-ax^2 + (2a-b)x + b - c)e^{-x}$ et $y_p'' = (ax^2 + (-4a+b)x + 2a - 2b + c)e^{-x}$

$y_p'' - 3\ y_p' + 2y_p = (6ax^2 - (10a - 6b)x + 2a - 5b + 6c)e^{-x} = (2x^2 - 3)e^{-x}$

$$\begin{cases} 6a = 2 \\ -10a + 6b = 0 \\ 2a - 5b + 6c = -3 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{3} \\ b = \frac{5}{9} \\ c = \frac{-4}{27} \end{cases} \Rightarrow y_p = \frac{1}{27}(9x^2 + 15x - 4)e^{-x}$$

Then : $y_G = C_1e^x + C_2e^{2x} + \frac{1}{27}(9x^2 + 15x - 4)e^{-x}$.

2) $(E) : y'' + 4y = 3\sin(2x)$

$i)$ $y_0 = ?$ Solution of (E_0) : $y'' + 4y = 0$ $P(r) = r^2 + 4 = (r - 2i)(r + 2i)$

Then : $y_0 = C_1\cos(2x) + C_2\sin(2x)$

$ii)$ $y_p = ?$ We have $f(x) = A\,cos(\beta x) + B\,sin(\beta x) = 3\sin(2x)$, $\beta = 2$

$\beta i = 2i$ is a root of P. Then: $y_p = x\big(A\,cos(2x) + B\,sin(2x)\big)$

$y_p' = A\,cos(2x) + B\,sin(2x) + x(-2A\sin(2x) + 2B cos(2x))$

$$y_p'' = -4A\,sin(2x) + 4B\,cos(2x) + x(-4Acos(2x) - 4Bsin(2x))$$

$$y'' + 4y = -4A\,sin(2x) + 4B\,cos(2x) = 3\sin(2x) \Rightarrow \begin{cases} -4A = 3 \\ 4B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{-3}{4} \\ B = 0 \end{cases}$$

Then : $y_p = \frac{-3}{4} x \cos(2x)$

Conclusion : $y_G = y_0 + y_p = \left(\frac{-3}{4}x + C_1\right)\cos(2x) + C_2\sin(2x)$.